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WIGNER FUNCTIONS OF A QUANTUM DAMPED OSCILLATOR

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The nonstationary quadratic quantum system which by considered as a quantum model of a damped oscillator is investigated in the framework of Wigner-Weyl representation. The explicite expressions of ordinary Wigner function and the smoothed one for this system are obtained.

In last few years Wigner-Weyl representation [1,2] plays a very important role in investigation of modern quantum mechanical problems [3-5]. With the use of Wigner function the quantum corrections to the classical results can be obtained [4]. This representation has been served as a very convenient tool for studing of quantum mechanical systems with quadratic Hamiltonians with respect to operators of coordinates and momenta [6-10]. Wigner function allows to find the average value of an arbitraty physical quantity but it is impossible to interpritate it as a probability because it can take the negative values. Smoothing the Wigner function with the different weight functions [11] the positive determening of it can be reached and the resulting function is known as smoothed Wigner function leads to the mistakes in the calculations of the average values in comparison with exact quantum mechanical results but in the classical limit just it begins to play an essential role [3].

In this work using the results [9, 10, 12] we consider in the framework of Wigner-Weyl representation a quantum system with a Hermitian nonstationary Hamiltonian:

$$\hat{H}(t)=\frac{1}{2}\left[\hat{p}^2 e^{-2\Gamma(t)}+\omega^2(t)\,e^{2\Gamma(t)}\hat{x}^2\right]-f(t)\,e^{2\Gamma(t)}\hat{x} \quad , \qquad (1)$$

with corresponding equations of the motion:

$$\begin{aligned}&\dot{x}=pe^{-2\Gamma(t)}\ ,\ \dot{p}=-\omega^2(t)\,e^{2\Gamma(t)}x+f(t)\,e^{2\Gamma(t)}\\ &\ddot{x}+2\Gamma(t)\,\dot{x}+\omega^2(t)\,x=f(t)\end{aligned} \quad , \qquad (2)$$

Therefore, the quantum system with such a Hamiltonian can be considered as a quantum analog of a classical damped forced harmonic oscillator with time-dependent parameters. Let as find the integrals of the motion of the system, i.e. operators $\hat{I}(t)$ satisfying the equation $\left[i\left(\partial/\partial t\right)-\hat{H},\hat{I}\right]=0\ (\hbar=1)$. For the quadratic Hamiltonian systems in one-dimensioanal case all integrals of the motion can be constructed from two independent linear once. For the present system one obtains two mutually Hermitian conjugate linear integrals of the motion satisfying the relation $\left[\hat{A}(t)\,\hat{A}^{+}(t)\right]=1$.

The operator has the form:

$$\hat{A}(t)=(i/\sqrt{2})\left[\varepsilon(t)\hat{p}-\dot{\varepsilon}(t)\,e^{2\Gamma(t)}\hat{x}\right]+\delta(t)/\sqrt{2} \qquad (3)$$

where

$$\delta(t)=-i\int\varepsilon(\tau)\,e^{2\Gamma(t)}f(\tau)\,d\tau$$

and $\varepsilon(t)$ is a complex function satisfying the equation:

$$\ddot{\varepsilon}+2\dot{\Gamma}\dot{\varepsilon}+\omega^{2}(t)\varepsilon=0 \qquad (4)$$

and additional relation:

$$e^{2\Gamma(t)}(\dot{\varepsilon}\varepsilon^{*}-\dot{\varepsilon}^{*}\varepsilon)=2i \qquad (5)$$

Let us consider the operator of the following form:

$$\hat{K}(t)=\hat{A}(t)\hat{A}^{+}(t)+1/2 \qquad (6)$$

It is easy to check that this operator is an integral of the motion and has a sence of the operator of quasi-particle number and its eigenfunctions satisfy the Schrödinger equation of this system:

$$K\psi_{n}=(n+1/2)\psi_{n} \qquad (7)$$

The eigenfunctions have a form

$$\psi_n(x,t) = (n!)^{1/2}\left(\frac{\varepsilon^*}{2\varepsilon}\right)^{n/2}(\pi\varepsilon^2)^{-1/4}\exp\left[\frac{i\dot{\varepsilon}}{2\varepsilon}e^{2\Gamma(t)}x^2 - \right.$$
$$\left. -\frac{x\delta}{\varepsilon} - \frac{(\varepsilon\delta)^*}{4\varepsilon} - \frac{1}{4}|\delta|^2 - \frac{i}{2}\int Im\,(\delta\dot{\delta}^*)\,d\tau\right]H_n\left(\frac{x + Re\,(\varepsilon^*\delta)}{|\varepsilon|}\right) \tag{8}$$

The corresponding Wigner function is :

$$W_n(p,q) = 2\,(-1)^n e^{-2z(t)}L_n(4z(t))\ , \tag{9}$$

where $L_n(x)$ are Lager polinomials and

$$z(t) = 1/2\left[|\varepsilon|^2 p^2 + |\dot{\varepsilon}|e^{4\Gamma(t)}q^2 - 2e^{2\Gamma(t)}Re\,(\dot{\varepsilon}\varepsilon^*)\,qp + \right.$$
$$\left. + Im\,(\varepsilon^*\delta)\,p - e^{2\Gamma(t)}Im\,(\dot{\varepsilon}^*\delta)\,q + |\delta|^2 + ie^{2\Gamma(t)}Re\,(\dot{\varepsilon}\varepsilon^*)\right]$$

For the determening of the smoothed Wigner function it is necessary to consider the "probability" phase particle hit into the finit domain $\Delta p\cdot\Delta q$ of the phase plane

$$U_{\Delta p\Delta q}(p,q) = \iint\limits_{\Delta p\Delta q} W\,(p+p_1,q+q_1)\,dp_1dq_1 \tag{10}$$

For the simplycity one considers instead of the exact boundary domaine the "spreaded" one by introducing the following quantity:

$$U_{\Delta p\Delta q}(p,q) = \iint\limits_{-\infty}^{\infty}\exp\left(-\frac{p_1^2}{2\Delta_p^2} - \frac{q_1^2}{2\Delta_q^2}\right)W\,(p+p_1,q+q_1)dp_1dq_1 \tag{11}$$

The smoothed Wigner function is given (assuming that $\Delta p\cdot\Delta q = 1/2$) by

$$\overline{W}_n(p,q) = U_{n\Delta p\Delta q}/\pi = (2^n n!\,\pi^{1/2}\Delta q|\varepsilon|)^{-1}\exp-\left[\frac{Re\,(\varepsilon\delta^*)}{2|\varepsilon|^2} - \right.$$
$$\left. -\frac{1}{2}|\delta|^2 - i\int Im\,(\dot{\delta}\delta^*)\,d\tau - \frac{q}{4\Delta_q^2} + Re\,(L^2S^{-1})\right]\times \tag{12}$$
$$\times\frac{|S|\left[\left(|\varepsilon|^2 - S^{-1}\right)\left(|\varepsilon|^2 - \left(S^*\right)^{-1}\right)\right]^{n/2}}{|\varepsilon|^n}\left|H_n\left[\frac{Re\,(\varepsilon^*\delta) + LS^{-1}}{\left(|\varepsilon|^2 - S^{-1}\right)^{1/2}}\right]\right|^2$$

where

$$L = \frac{q}{2\Delta_q^2} - \frac{\delta}{\varepsilon} - ip\,,\; S = \frac{1}{2\Delta_q^2} - \frac{i\dot{\varepsilon}}{\varepsilon} e^{2\Gamma(t)}$$

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SÖNƏN HARMONİK OSSİLYATORUN VİQNER FUNKSİYASI

Viqner-Veyl təsvirində sönən harmonik ossilyator üçün kvant modeli ola biləcək kvadratik qeyri stasionar kvant sistemi tədqiq edilmişdir. Bu sistem üçün adi hamarlanmış Viqner funksiyasının aşkar şəkli tapılmışdır.

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ФУНКЦИИ ВИГНЕРА ЗАТУХАЮЩЕГО КВАНТОВОГО ОСЦИЛЛЯТОРА

Рассмотрена в рамках представления Вигнера-Вейля квадратичная нестационарная квантовая система, которая может служить квантовой моделью затухающего гармонического осциллятора. Получены явные выражения для обычной и сглаженной функций Вигнера этой системы.