УДК 530.145; 539.12

ALGEBRAIC TREATMENT OF A FINITE-DIFFERENCE EQUATION FOR A *N*-DIMENSIONAL RELATIVISTIC COULOMB PROBLEM

SH.M.NAGIYEV
Institute of Physics, Academy of Sciences of Azerbaijan
H.Gavid str.33, Baku 370143
(Received 16.12.94)

An algebraic solution of a Coulomb problem in the *N*-dimensional relativistic configuration $\vec{r}_N$ - space is presented.

1.Introduction

In the papers [1-6] in the framework of quasipotential approach [7,8] a version of the relativistic quantum mechanics has been developed. It is based on the concept of the relativistic 3-dimensional configuration $\vec{r}$-space [1], which arises when we use the Fourier expansion over the relativistic "plane waves" [9] $\left(\frac{h}{2\pi} = m = c = 1\right)$

$$\langle\vec{r}|\vec{p}\rangle = (p_0 - \vec{p}\vec{n})^{-1-ir}$$
$$\vec{r} = r\vec{n},\ \ 0 < r < \infty,\ \ \vec{n}^2 = 1 \tag{1.1}$$

instead of the usual plane waves $e^{i\vec{p}\vec{r}}$. The system of functions (1.1) is a complete and orthogonal one on the mass hyperboloid $p_0^{\ 2} - \vec{p}^2 = 1$ of a unit mass particle. From the geometrical point of view, on the mass hyperboloid $p_0^{\ 2} - \vec{p}^2 = 1$ the 3-dimensional Lobachevsky space is realized, the group of motions of which is the Lorentz group *SO (3,1)*.

In the nonrelativistic limit

$$\lim_{c\to\infty} \langle\vec{r}|\vec{p}\rangle = e^{i\vec{p}\vec{r}} \tag{1.2}$$

The relativistic quantum mechanics in $\vec{r}$-space carries many important features of quantum mechanics, but in distinction from quantum mechanics the equation for the wavefunction of the relativistic motion is written in a finite-difference form. Thus the relativistic generalizations of some exactly solvable problems of quantum mechanics (harmonic oscillator, square-well and Coulomb problem) has been considered [2-6].

There are also the algebraic approaches in the description of quantum systems by the theory of dynamical symmetries [10,11], potential groups [12], Casimir operators with mixed spectrum [13].

The purpose of the present paper is to solve a finite-difference equation that describes a Coulomb problem in a N-dimensional relativistic configuration space by Lie algebraical method.

2. N - dimensional relativistic configuration $\vec{r}_N$-space

The N-dimensional relativistic configuration $\vec{r}_N$-space is introduced [14] by analogy with the 3-dimensional relativistic configuration $\vec{r}$-space. Namely, transition to the $\vec{r}_N$-space

$$\psi(\vec{r}_N) = \frac{1}{(2\pi)^{N/2}} \int \frac{d\vec{p}_N}{p_0} \langle \vec{r}_N | \vec{p}_N \rangle \psi(\vec{p}_N) \tag{2.1}$$

is performed by the expansion in terms of the matrix elements

$$\langle \vec{r}_N | \vec{p}_N \rangle = (p_0 - \vec{p}_N \vec{n}_N)^{-\frac{N-1}{2} - ir}$$
$$\vec{r}_N = r\vec{n}_N, \quad \vec{n}_N^{\,2} = 1 \tag{2.2}$$

of the infinite - dimensional unitary representations of the N - dimensional momentum Lobachevsky space motion group $SO(N,1)$, which is realized on the upper sheet of the hyperboloid $p_0^2 - \vec{p}_N^{\,2} = p_0^2 - p_1^2 - p_2^2 - \cdots - \vec{p}_N^{\,2} = 1$.

The N-dimensional relativistic "plane waves" (2.2) are the eigenfunctions of the free Hamiltonian $H_0^{(N)}$, which in the spherical system of coordinates $\vec{r}_N = (r, \theta_1, \theta_2, \ldots, \theta_{N-1})$ has the form

$$H_0^{(N)} = \mathrm{ch}\, i\partial_r + \frac{i(N-1)}{2r} \mathrm{sh}\, i\partial_r - \frac{\Delta_\theta}{r[2r - i(N-3)]} e^{i\partial_r} \tag{2.3}$$

where Δ_θ is the Laplace operator on the unit sphere S^{N-1}. The eigenfunctions of Δ_θ are the N-dimensional spherical harmonics $Y_{Nl}(\theta_1, \theta_2, \ldots, \theta_{N-1})$:

$$\Delta_\theta Y_{Nl}(\theta_1, \theta_2, \ldots, \theta_{N-1}) = -l\,(l + N - 2)\, Y_{Nl}(\theta_1, \theta_2, \ldots, \theta_{N-1}) \;, \tag{2.4}$$

3. Coulomb problem in the *N*-dimensional relativistic configuration $\vec{r}_N$-space

Here we consider a simple generalization of the 3-dimensional relativistic Coulomb problem to *N*-dimensional case. In the *N*-dimensional relativistic configuration $\vec{r}_N$-space a Coulomb problem is described by the finite-difference equation

$$\left(H_0^{(N)} - \alpha/r\right)\psi(\vec{r}_N) = E\psi(\vec{r}_N) \ , \tag{3.1}$$

Due to the geometrical *O(N)* symmetry the angular dependence of the wavefunction is defined by the spherical harmonics

$$\psi(\vec{r}_N) = \left[(-r)^{\left(\frac{N-1}{2}\right)}\right]^{-1} \psi_{Nl}(r)\, Y_{Nl}(\theta_1,\theta_2,\ldots,\theta_{N-1}) \ . \tag{3.2}$$

Therefore, a problem is reduced to finding the eigenvalues and eigenfunctions of the radial part of a Hamiltonian

$$\left(H_0^{(N)\,rad} - \alpha/r\right)\psi_{Nl}(r) = E\psi_{Nl}(r) \tag{3.3}$$

where

$$H_0^{(N)\,rad} = ch\, i\,\partial_r + \frac{L(L+1)}{2r^{(2)}}e^{i\partial_r} \quad , \quad L = l + \frac{N-3}{2} \tag{3.4}$$

and $x^{(\lambda)}$ is the generalized degree $x^{(\lambda)} = i^{\lambda}\dfrac{\Gamma(-ix+\lambda)}{\Gamma(-ix)}$. To solve the equation (3.1) algebraically, we introduce three operators

$$\begin{aligned} \Gamma_0 &= r\,ch\, i\,\partial_r + \frac{i\,(N-1)}{2} sh\, i\,\partial_r - \frac{\Delta_0}{2r - i\,(N-3)}e^{i\partial_r} \ , \\ \Gamma_4 &= r \ , \\ \mathrm{T} &= r\,sh\, i\,\partial_r + \frac{i\,(N-1)}{2} ch\, i\,\partial_r - \frac{\Delta_0}{2r - i\,(N-3)}e^{i\partial_r} \end{aligned} \tag{3.5}$$

They satisfy the commutation relations of Lie algebra *SO (2,1)*:

$$[\Gamma_0,\Gamma_4] = iT \ , \ [\Gamma_4,\mathrm{T}] = -i\Gamma_0 \ , \ [\mathrm{T},\Gamma_0] = i\Gamma_4 \tag{3.6}$$

The direct calculation shows that the Casimir operator is equal to

$$C_2 = \Gamma_0^2 - \Gamma_4^2 - T^2 = -\Delta_0 + \frac{(N-1)\,(N-3)}{4} = L\,(L+1)\,, \qquad (3.7)$$

From (3.7) it is follows that to each value of the orbital quantum number corresponds an irreducible unitary representation $D+(-L-1)$ of the group *SO(2,1)* with the eigenvalues of the compact generator Γ_0, which are bounded below and are equal to $n = n_r + L + 1$, where $n_r = 0, 1, 2, \ldots$ is the radial quantum number.

In terms of (3.5) the equation (3.1) (after multiplying by r) has the following form

$$(\Gamma_0 - E\Gamma_4 - \alpha)\psi = 0. \qquad (3.8a)$$

By means of the unitary transformation $\tilde{\psi} = e^{-i\Theta T}\psi$, the eigenvalue equation (3.8a) can be written as

$$(a\Gamma_0 + b\Gamma_4 - \alpha)\tilde{\psi} = 0\ , \qquad (3.8b)$$

where $a = ch\Theta - esh\Theta,\ \ b = sh\Theta - Ech\Theta.$

Let us consider the cases $|E| < 1$ and $|E| > 1$ in (3.8b) separalely.

If $|E| < 1$, then the compact generator Γ_0 can be diagonalized. Choosing the angle Θ in (3.8b) as $\Theta = \frac{1}{2} ln \frac{1-E}{1+E}$, we reduce (3.8b) to the form

$$\left(\sqrt{1-E^2}\,\Gamma_0 - \alpha\right)\tilde{\psi} = 0 \qquad (3.9)$$

From (3.9) we obtain a discrete energy spectrum for the *N*-dimensional relativistic Coulomb problem

$$E_n = \pm\sqrt{1 - \frac{\alpha^2}{n^2}}\ \ ,\ \ n = n_r + 1 + \frac{N-1}{2} \qquad (3.10)$$

If $|E| > 1$, then the noncompact generator Γ_4 can be diagonalized. For this purpose choose $\Theta = \frac{1}{2} ln \frac{1-E}{1+E}$ and obtain instead of (3.8b) the equation

$$\left[\sqrt{E^2-1}\,(sgnE)\Gamma_4 + \alpha\right]\tilde{\psi} = 0 \qquad (3.11)$$

Since Γ_ζ has the continuous real spectrum $\lambda \in R$, from (3.11) we find the continuous spectrum of the system

$$E_\lambda = \pm\sqrt{1+\alpha^2/\lambda^2} \tag{3.12}$$

Now we can also define the explicit form of the radial wavefunctions algebraically. They are expressed in terms of hypergeometric function $F(a,b;c;z)$.

In the case of the discrete spectrum, $E_n = \cos\chi_n$, we have

$$\psi_{Nnl}(r) = (-r)^{(L+1)} e^{-r\chi_n} F(-n_r, -ir+L+1; 2L+2; 1-e^{-2i\chi_n})$$

and in the case of the continuous spectrum, $E_n = ch\chi_n$, we have

$$\psi_{N\chi l}(r) = (-r)^{(L+1)} e^{ir\chi} F\left(L+1\frac{i\alpha}{sh\chi}, -ir+L+1; 2L+2; 1-e^{-2\chi}\right)$$

Reference

1. *Kadyshevsky V.G., Mir-Kasimov R.M., Skachkov N.B.*,Nuovo Cimento, 1968, v. 55, p. 233.
2. *Kadyshevsky V.G.,Mir-Kasimov R.M.,Freeman M.*, Jad. Fizika, 1969, v. 9, p. 646.
3. *Freeman M., Mateev M.D.,Mir-Kasimov R.M.*, Nucl. Phys. B, 1969, v.12, p. 197.
4. *Donkov A.D. et al.* Teor.Mat.Fizika, 1971, v. 8, p. 61.
5. *Atakishiyev N.M.,Mir-Kasimov R.M.,Nagiyev Sh.M.*, Ann. Phys. (Leipzig), 1985, v. 42, p. 25.
6. *Nagiyev Sh.M.*, Teor. Mat. Fizika, 1989, v. 80, p. 40.
7. *Logunov A.A., Tavkhelidze A.N.*, Nuovo Cimento, 1963, v. 29, p. 380.
8. *Kadyshevsky V.G.*, Nucl. Phys. B, 1968, v. 6, p. 125.
9. *Shapiro I.S.*, Sov. Phys. Doklady, 1956, v. 1, p. 91.
10. *Barut A.O.*, Dynamical Groups and Generalized Symmetries in Quantum Theory (University of Conterbury, Christchurch, New Zealand, 1972).
11. *Malkin I.A., Manko V.I.*, The Dynamical symmetries and Coherent states of quantum systems (Nauka, Moccow, 1979).
12. *Alhassid Y., Gursey F., Iachello F.*, Ann. Phys., 1983, v. 148, p. 346.
13. *Frank A., Wolf K.B.*, Phys.Rev.Lett., 1984, v. 52, p. 1717.
14. *Atakishiyev N.M.*, Preprint Inst. Physics Azerbaijan Acad. Sci. 1986, № 201, p. 9.

Ş.M. Nağıyev

N-ÖLÇÜLÜ RELYATİVİSTİK KULON MƏSƏLƏSİ ÜÇÜN SONLU-FƏRQ TƏNLİYİNƏ CƏBRİ YANAŞMA

N-ölçülü relyativistik $\vec{r}_N$-konfigurasiya fəzasında Kulon məsələsinin cəbri həlli verilmişdir.

Ш.М. Нагиев

АЛГЕБРАИЧЕСКИЙ ПОДХОД К КОНЕЧНО-РАЗНОСТНОМУ УРАВНЕНИЮ ДЛЯ *N* - МЕРНОЙ РЕЛЯТИВИСТСКОЙ КУЛОНОВСКОЙ ЗАДАЧИ

Дано алгебраическое решение кулоновской задачи в N-мерном релятивистском конфигурационном $\vec{r}_N$-пространстве.