ON NEW ADDITION THEOREM FOR THE BESSEL FUNCTIONS

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Addition theorem for the Bessel functions used for evaluation of higher-order Feynman diagrams, is obtained by group theoretical methods.

Connection between group representations and special functions are well known since the work of Vilenkin [1], Miller [2], and these two fields are progressing together as it can be nicely seen in this Talman [3] book based on Wigner lectures. The theory of group representations gives new addition theorems for special functions.

Some years ago we outlined [4-6] the general procedure of evaluating Feynman integrals, in which the spins of the particles are used as regulating parameters. In this context, we established a new addition theorem for the Bessel functions, which can be used to evaluate the higher-order diagrams such as triangle graph and multiple-loop diagrams. The present paper is devoted to the derivation of the addition theorem by group theoretic methods. For this purpose we will have to deal with the universal covering group $\overline{E}_4$ of the 4-dimensional Euclidean group E_4 which is semi-direct product of the 4-dimensional translation group T_4 and $SU_2\times SU_2$.

An element of E may be written as $(\underset{\sim}{x}, (u_1, u_2))$, where u_1, u_2, SU_2, and $\underset{\sim}{x}$ is a two-by-two matrix constructed from the translation 4-vectors x as

$$\underset{\sim}{x} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 + i x_4\sigma_4 \qquad (1)$$

Here σ_4 is the unit matrix and σ_k, $k=1,2,3,$ are the familiar Pauli matrices. The unit element is $(0,(1,1))$. The group multiplication is defined by

$$(\underset{\sim}{x}, (u_1, u_2))(x_1', (u_1', u_2')) = = (\underset{\sim}{x} + u_1', \underset{\sim}{x}', u_2^t, (u_1u_1' \cdot u_2u_2')) \qquad (2)$$

(u^t being the transposed of u).

The unitary irreducible representations of $\overline{E}_4$ characterized by the real number $\rho>0$ may be realized in the Hilbert space $L^2(SU_2)$ of Lebesgue square integrable functions over SU_2 with the inner product

$$(f,g) = \int_{SU_2} f(u)\, g(u)\, du \quad f,g\in L^2(SU_2) \qquad (3)$$

where du is the normalized invariant measure on SU_2. The unitary operator $U^\rho(\underset{\sim}{x}, (u_1, u_2))$ representing the group element $(\underset{\sim}{x}, (u_1, u_2))$ acts on $f(u)$ in the following way

$$U^\rho(\underset{\sim}{x}, (u_1, u_2))\, f(u) = = \exp\left\{\frac{1}{2} i\rho\, tr\left(\underset{\sim}{x} u^{-1}\right)\right\} f(u_1 u u_2^t) \qquad (4)$$

where tr denotes the trace of the matrix indicated.

An orthonormal basis for $L^2(SU_2)$ is known as [1]

$$h^j_{mn}(u) = \sqrt{2j+1}\, D^j_{mn}(u)\,, \quad j=0,\frac{1}{2},1,\ldots,\; m,n=-j,-j+1,\ldots j, \qquad (5)$$

where $D^j(u)$ are the irreducible representation matrices of SU_2.

In the following we consider the matrix element of the representation (4) with respect to the orthonormal basis given in (5). These matrix elements may be calculated from the formula

$$\langle j_1 m_1 n_1 | U^\rho(\underset{\sim}{x}, (u_1, u_2)) | jmn\rangle = = (h^{j_1}_{m_1 n_1}, U^\rho(\underset{\sim}{x}, (u_1, u_2))\, h^j_{mn}) = = \sqrt{(2j+1)(2j_1+1)} \int_{SU_2} \exp\left\{\frac{1}{2} i\rho\, tr\left(\underset{\sim}{x} u^{-1}\right)\right\} * * D^{j_1}_{m_1 n_1}(u)\, D^j_{mn}(u_1 u u_2^t)\, du \qquad (6)$$

It is sufficient to calculate only matrix elements of the translation subgroup. In order to simplify the calculations one may use formula

$$\exp\left\{\frac{1}{2} i\rho\, tr\left(\underset{\sim}{x} u^{-1}\right)\right\} = = 2(\rho|x|)^{-1} \sum_{j=0,\frac{1}{2},1}^{\infty} i^{2j}(2j+1)\, J_{2j+1}(\rho|x|)\, tr D^j(\hat{\underset{\sim}{x}} u^{-1}) \qquad (7)$$

which follow from 7.10 (5) of [7]. Here $\hat{\underset{\sim}{x}}$ is two-by-two matrix constructed from the unit vector $\hat{x}$ corresponding to x and J_{2j+1} is the Bessel functions of the first kind [7]. Making use of (7) and integrating with respect to u we obtain

$$<j_1m_1n_1| U^\rho(\underset{\sim}{x}, (1,1)) | jmn> =$$
$$= 2\left(\frac{2j+1}{2j_1+1}\right)^{\frac{1}{2}} (\rho|x|)^{-1} \sum_{j_2=|j_1-j|}^{j_1+j} \sum_{m_2,n_2=-j_2}^{j_2} i^{2j_2}(2j_2+1)\times$$
$$\times\langle j_1m_1j_2m_2|jm\rangle\langle j_1n_1j_2n_2|jn\rangle D^{j_2}_{m_2n_2}(\hat{\underset{\sim}{x}})\, J_{2j_2+1}(\rho|x|) \quad (8)$$

where $<j_1m_1j_2m_2|jm>$ are Clebsch-Gordan coefficients of SU_2.

In particular,

$$<000| U^\rho(\underset{\sim}{x}, (1,1)) | jmn> =$$
$$=2\sqrt{2j+1}\, i^{2j} (\rho|x|)^{-1} D^j_{mn}(\hat{\underset{\sim}{x}})\, J_{2j+1}(\rho|x|) \quad (9)$$

By the group property

$$U^\rho(\underset{\sim}{x}-\underset{\sim}{x}', (1,1)) = U^\rho(\underset{\sim}{x}, (1,1))\, U^\rho(-\underset{\sim}{x}', (1,1))$$

we obtain a new addition theorem for the Bessel function of the first kind

$$\frac{1}{|x-x'|} D^j_{mn}(\widehat{\underset{\sim}{x-x'}})\, J_{2j+1}(\rho|x-x'|) =$$
$$= 2(2j+1)^{-1}\frac{1}{\rho|x||x'|} \sum_{j_1,j_2} \sum_{m_1,n_1=-j_1}^{j_1} \sum_{m_2,n_2=-j_2}^{j_2} (-)^{j_1-j_2-j} \times$$
$$\times (2j_1+1)(2j_2+1)\langle j_1m_1j_2m_2|jm\rangle\langle j_1n_1j_2n_2|jn\rangle\times$$
$$\times D^{j_1}_{m_1n_1}(\hat{\underset{\sim}{x}})\, D^{j_2}_{m_2n_2}(\hat{\underset{\sim}{x}}')\, J_{2j_1+1}(\rho|x|)\, J_{2j_2+1}(\rho|x'|) \quad (10)$$

Using the formulas 8.5(12) and 8.11(15) of [8] we can also get an addition theorem for the modified Bessel function of the third kind K_{2j+1} [7] from (10).

$$\frac{1}{|x-x'|} D^j_{mn}(\widehat{\underset{\sim}{x-x'}})\, K_{2j+1}(\rho|x-x'|) =$$
$$= 2(2j+1)^{-1}\frac{1}{\rho|x||x'|} \sum_{j_1,j_2} \sum_{m_1,n_1=-j_1}^{j_1} \sum_{m_2,n_2=-j_2}^{j_2} (2j_1+1)\times$$
$$\times (2j_2+1)\langle j_1m_1j_2m_2|jm\rangle\langle j_1n_1j_2n_2|jn\rangle\times$$
$$\times D^{j_1}_{m_1n_1}(\hat{\underset{\sim}{x}})\, D^{j_2}_{m_2n_2}(\hat{\underset{\sim}{x}}')\, I_{2j_1+1}(\rho|x|)\, K_{2j_2+1}(\rho|x'|)$$

if

$$|x'| > |x|.$$

Here I_{2j+1} is the modified Bessel function of the first kind. In fact, we have the addition theorem for covariant propagator [4] for massive particle of spin j in a $2j+1$ –component formalism.

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BESSEL FUNKSİYALARININ YENİ TOPLAMA TEOREMİ HAQQINDA

Yüksək tərtibli Feynman diaqramlarının hesablanmasında tətbiq olunmuş Bessel funksiyalarının toplama teoremi qrupp nəzəriyyəsi metodları ilə isbat edilmişdir.

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О НОВОЙ ТЕОРЕМЕ СЛОЖЕНИЯ ДЛЯ ФУНКЦИЙ БЕССЕЛЯ

Теорема сложения для функций Бесселя, использованная при вычислениях фейнмановских диаграмм высокого порядка, доказана с помощью группового метода.

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