ENERGY SPECTRUM OF q-ANALOGUE OF THE RELATIVISTIC HYDROGEN ATOM

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The $so_q(4)$ quantum algebra is used for the description of a q-analogue of the relativistic hydrogen atom. The discrete spectrum of q-analogue of the relativistic hydrogen is obtained algebraically.

1. In connection with the developments of the quantum groups and algebras in the last years the q-deformation of the various physical models attracts physicist's attention. The examples of description of the symmetry properties of the physical models using the quantum algebras were given. In particular, in papers [1-8] the problems of construction of the q-deformed quantum oscillators were discussed and the explicit realizations for the symmetry algebras $su_q(2)$ and $su_q(1.1)$ were given. In [9,10] the motion of the classical and quantum q-deformed free particles was studied. The papers [11,12] are devoted to the discussion of the properties of the q-analogue of the nonrelativistic hydrogen atom. In this papers a discrete spectrum of the q-hydrogen atom has been found algebraically:

$$E_q = -\frac{me^4}{2h^2\left\{\frac{1}{2}\sum_{i=1}^{4}([n_i+1]+[n_i])/2\right\}^2} \qquad (1.1)$$

where m is the reduced mass of the atom, $n_i=0,1,2,3,...$ and

$$[x] = \frac{q^x - q^{-x}}{q-q^{-1}} \qquad (1.2)$$

is the q-analogue of the number x. The quantum numbers n_i in (1.1) satisfy the constraint condition $[n_1]+[n_2]=[n_3]+[n_4]$. Limit $q\to 1$ (1.1) coincides with the spectrum of the ordinary nonrelativistic hydrogen atom

$$E_n = -\frac{me^4}{2h^2n^2}, \quad n = n_1+n_2+1 = n_3+n_4+1. \qquad (1.3)$$

The purpose of the present work is to introduce q-analogue of the three-dimensional relativistic hydrogen atom [13]. We shall deal here cigth the discrete spectrum of the relativistic hydrogen atom and shall obtain its q-analogue by passing from the invariance algebra $so(4)\sim su(2)\times su(2)$ of the ordinary relativistic hydrogen atom system [14] to its q-analogue $so_q(4)\sim su_q(2)\times su_q(2)$.

2. We review some facts about the quantum algebra $su_q(2)$. The algebra $su_q(2)$ is defined by the following commutation relations:

$$[J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2J_z] \qquad (2.1)$$

The irreducible representations of $su_q(2)$ are finite dimensional and have dimension $2j+1$, where the weight j takes the values $j=0,1/2,1,3/2,...$. As follows from (2.1), the action of the operators of the q-angular momentum $J_\pm, J_z$ in an arbitrary irreducible representation space $\{\psi_{jm}\}$, $-j\le m\le j$, are defined by the following formulas:

$$J_z\psi_{jm}=m\psi_{jm}, \; J_\pm\psi_{jm}=\sqrt{[j\mp m][j\pm m+1]}\,\psi_{jm\pm1} \qquad (2.2)$$

For the Casimir operator $\vec{J}^2 = J_-J_+ + [J_z][J_z+1]$ we obtain

$$\vec{J}^2\psi_{jm} = [j][j+1]\psi_{jm} \qquad (2.3)$$

In the limit $q\to 1$ we have $[x]\to x$, and therefore in this limit the formulas (2.1)-(2.3) become identical to the well known expressions for the algebra $su(2)$.

3. It is known that [14] the finite-difference Hamiltonian

$$H = H_0 - \alpha/r, \quad H_0 = mc^2 ch\left[\frac{ih}{mc}\frac{\partial}{\partial r}\right] + \frac{ihc}{r} sh\left[\frac{ih}{mc}\frac{\partial}{\partial r}\right] + \frac{\vec{L}^2}{2mr^2} e^{\frac{ih}{mc}\frac{\partial}{\partial r}} \qquad (3.1)$$

for the three-dimensional relativistic hydrogen atom [13] commutes with the orbital angular momentum $\vec{L}$ and the Runge-Lentz vector $\vec{A}$. Further, $\vec{L}$ and $\vec{A}$ satisfy the following relation

$$\vec{L}\vec{A} = \vec{A}\vec{L} = 0 \qquad (3.2)$$

In the case of $E<0$ (the discrete spectrum) the components of $\vec{L}$ and $\vec{A}$ generate the Lie algebra $so(4)$. By introducing following two independent angular momentum operators

$$\vec{J}^{(1)} = \frac{1}{2}(\vec{L}+\vec{A}), \; \vec{J}^{(2)} = \frac{1}{2}(\vec{L}-\vec{A}) \qquad (3.3)$$

the Lie algebra $so(4)$ can be rewritten as $su(2)\times su(2)$, namely

$$[J_z^{(a)},J_\pm^{(a)}]=\pm J_\pm^{(a)},\ [J_+^{(a)},J_-^{(a)}]=2J_z^{(a)},\ a=1,2. \quad (3.4)$$

In accordance with the relations (3.2) we have

$$(\vec{J}^{(1)})^2=(\vec{J}^{(2)})^2=\frac{1}{4}(\vec{L}^2+\vec{A}^2)=h^2 j(j+1). \quad (3.5)$$

The Hamiltonian (3.1) can be expressed by the vectors $(\vec{J}^{(1)})^2$ and $(\vec{J}^{(2)})^2$ as follows

$$\left[1-\frac{H^2}{m^2c^4}\right]\left[2(\vec{J}^{(1)})^2+2(\vec{J}^{(2)})^2+h^2\right]=\frac{\alpha^2}{c^2} \quad (3.6)$$

Thus, from equations (3.5) and (3.6), it can recover the discrete spectrum of H:

$$E_j=mc^2\sqrt{1-\frac{\alpha^2}{(4j(j+1)+1)^2h^2c^2}} \quad (3.7)$$

The expression (3.7) possesses the correct non-relativistic limit $E_j-mc^2\to-\frac{m\alpha^2}{2n^2h^2}$, where $n=2j+1$.

We can define q-deformed relativistic hydrogen atom with the Hamiltonian operator $\hat{H}_q$. For the q-analogue of the relativistic hydrogen atom, we also require that equations (3.2) are given by the following relation

$$\vec{L}_q\vec{A}_q=\vec{A}_q\vec{L}_q=0 \quad (3.8)$$

The components of $\vec{L}_q$ and $\vec{A}_q$ generate the quantum algebra $so_q(4)$. By introducing the following two q-angular momentum operators:

$$\vec{J}_q^{(1)}=\frac{1}{2}(\vec{L}_q+\vec{A}_q),\ \vec{J}_q^{(2)}=\frac{1}{2}(\vec{L}_q-\vec{A}_q) \quad (3.9)$$

the algebra $so_q(4)$ can be rewritten as $su_q(2)\times su_q(2)$, namely

$$[J_{qz}^{(a)},J_{q\pm}^{(a)}]=\pm J_{q\pm}^{(a)},\ [J_{q+}^{(a)},J_{q-}^{(a)}]=[2J_{qz}^{(a)}],\ a=1,2 \quad (3.10)$$

To realize (3.8) and (3.9), we consider a set of four independent q-harmonic oscillator's operators of creation, annihilation $a_{iq}^{\pm}$ and number operators N_{iq}, $i=1,2,3,4$, satisfying to the following expression:

$$a_{iq}^{-}a_{iq}^{+}-q^{-1}a_{iq}^{+}a_{iq}^{-}=q^{N_{iq}},\ [N_{iq},a_{iq}^{\pm}]=\pm a_{iq}^{+} \quad (3.11)$$

Then we have

$$J_{q+}^{(1)}=a_{1q}^{+}a_{2q}^{-},\ J_{q-}^{(1)}=a_{2q}^{+}a_{1q}^{-},\ 2J_{qz}^{(1)}=N_{1q}-N_{2q},$$
$$J_{q+}^{(2)}=a_{3q}^{+}a_{4q}^{-},\ J_{q-}^{(2)}=a_{3q}^{+}a_{4q}^{-},\ 2J_{qz}^{(2)}=N_{3q}-N_{4q}. \quad (3.12)$$

Next we extend the equation (3.6) to the quantum algebra $su_q(2)\times su_q(2)$, i.e. we assume that

$$\left[1-\frac{\hat{H}_q^2}{m^2c^4}\right]\left\{2(\vec{J}_q^{(1)})^2+2(\vec{J}_q^{(2)})^2+h^2\right\}=\frac{\alpha^2}{c^2} \quad (3.13)$$

From (3.13) we can obtain the energy of q-analogue of the relativistic hydrogen atom

$$E_{qj}=mc^2\sqrt{1-\frac{\alpha^2}{(4[j][j+1]+1)^2h^2c^2}} \quad (3.14)$$

It is readily verified that the expression (3.11) has correct limits in the cases of $q\to1$ and $c\to\infty$.

The q-deformed relativistic hydrogen atom defined by (3.14) has the same ground energy level as the ordinary relativistic hydrogen atom (3.7), what corresponds to the limiting situation $q=1$. Further more, its discrete spectrum exhibits the same degeneracy as the ordinary relativistic atom. The only difference between the cases $q=1$ and $q<>1$ arises in the position of the excited leuches (cf. [11]).

In conclusion we remark that, as in the nonrelativistic case [11, 12], it will be defined a q-deformed relativistic hydrogen atom via the Kustaanheimo-Stiefel transformation [15].

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RELYATİVİSTİK HİDROGEN ATOMUNUN q-ANALOQUNUN ENERJİ SPEKTRİ

Kvant cəbrindən $so_q(4)$ relyativistik hidrogen atomunu təsvir etmək üçün istifadə olunur. Relyativistik hidrogen atomunun q-analoqunun diskret spektri cəbri yolla tapılmışdır.

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ЭНЕРГЕТИЧЕСКИЙ СПЕКТР q-АНАЛОГА РЕЛЯТИВИСТСКОГО АТОМА ВОДОРОДА

Квантовая алгебра $so_q(4)$ используется для описания q-аналога релятивистского атома водорода. Дискретный спектр q-аналога релятивистского атома водорода получен алгебраически.

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