EXCITATION OF A RELATIVISTIC QUANTUM LINEAR OSCILIATOR BY HOMOGENECUS FIELD. I

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The quantum transition probability amplitudes S_{mn} from the stationary state n into the stationary state m for the relativistic linear oscillator under suddenly changed homogeneous field are calculated. It is shown that the quantities S_{mn} are expressed by the Meixner polynomials $M_n\ (m;\beta,c)$ of a discrete variable. The generating function for them is obtained. The nonrelativistic limit of S_{mn}, which gives a solution of the corresponding nonrelativistic problem, is investigated.

1. A relativistic quantum mechanics, developed in the papers [1-9], carries many important features of nonrelativistic quantum mechanics. It is based on the concept of relativistic configurational $\vec{r}$ - representation [1]. The relativistic $\vec{r}$ representation is introduced by the expansion of the wave function over the complete and orthogonal system of functions

$$\xi(\vec{p},\vec{r})=\left(\frac{P_0-\vec{p}\vec{n}}{mc}\right)^{-1-ir\frac{mc}{\hbar}},\qquad (1.1)$$

$$\vec{r}=r\vec{n},\ \vec{n}^2=1,\ P_0=\sqrt{\vec{p}^2+m^2c^2}.$$

They realize a basis for the irrefucible unitary representation of the Lorentz group. In the nonrelativistic limit we have $\xi(\vec{p},\vec{r})\rightarrow exp\,(i\vec{p}\vec{r}/\hbar)$.

An annologue of the Schrodinger equation in the relativistic cofigurational $\vec{r}$ - representation is a finite-difference quasipotential equation

$$[H_0+V(\vec{r})]\psi(\vec{r})=E\psi(\vec{r}),$$

where the free Hamiltonian has the from:

$$H_0=mc^2ch\left(i\frac{\hbar}{mc}\frac{\partial}{\partial r}\right)+\frac{i\hbar c}{r}sh\left(i\frac{\hbar}{mc}\frac{\partial}{\partial r}\right)+\frac{\vec{L}^2}{2mr^2}e^{i\frac{\hbar}{mc}\frac{\partial}{\partial r}}\qquad (1.2)$$

The relativistic generalizations to the case of the equation (1.2) of the solvable problems of the nonrelativistic quantum mechanics have been considered.

The aim of the present paper is to calculate and analyse the quantum transition amplitudes between the discrete spectrum states of a relativistic linear oscillator under suddenly changed homogeneous field.

2. There exits [5-9] the various models of the relativistic linear oscillator. We shall consider a model with the Hamiltonian

$$H^{osc}=H_0+V(x)=H_0+\frac{m\omega^2}{2}x\left(x+i\frac{\hbar}{mc}\right)e^{i\frac{\hbar}{mc}\frac{d}{dx}}\qquad (2.1)$$

and the same dynamical symmetry group SU (1.1) as in the nonreletivistic case. In the one dimensional case the free Hamiltonian H_0 and the momentum $\hat{P}_x$ operators have the form

$$H_0=mc^2ch\left(i\frac{\hbar}{mc}\frac{d}{dx}\right),\ \hat{P}_x=-mc\,sh\left(i\frac{\hbar}{mc}\frac{d}{dx}\right).$$

Their common eigenfunction is the relativistic one-dimensional plane wave

$$\xi(p,x)=\left(\frac{P_0-p}{mc}\right)^{-i\frac{mc}{\hbar}x}\qquad (2.2)$$

The generators of the dynamical symmetry group SU(1.1) for the given model are defined as follows

$$K_0=\frac{1}{\hbar\omega}H^{osc},\ K_1=\frac{mc}{\hbar}x,\qquad (2.3)$$
$$K_2=-\frac{c}{\hbar\omega}\left(\hat{P}_x-\frac{1}{c}V(x)\right)$$

and satisty the commutation relations

$$[K_0,K_1]=iK_2,\ [K_2,K_0]=iK_1,\ [K_1,K_2]=-iK_0.\qquad (2.4)$$

In the paper [10] the relativistic model of the linear oscillator (2.1) in an external homogeneous field $V_g(x)=gx$, which corresponds to a constant force $F(x)=-\frac{\partial V_g}{\partial x}=-g$, has been considered. It is descriebed by the equation

$$H^g\psi^g(x)\equiv[H^{osc}+gx]\psi^g(x)=E\psi^g(x).\qquad (2.5)$$

It was shown that, in comparison with the nonrelativistic case there exists a possibility of the discrete $|g|<mc\omega$ as well as continuous $|g|\geq mc\omega$ energy spectrum due to the relation between the value of th force and frequence in the present case. The solutions of the eq. (2.5), corresponding to the discrete energy spectrum $E_n=\hbar\omega(n+\nu)\sin\varphi_g,\ n=1,2,3,...,$ are expressed by the Meixner-Pollaczek polynomials

$$\psi_n^g(x) = c_n^g\left(\frac{\hbar\omega}{mc^2}\right)^{i\tilde{x}}\Gamma(\nu + i\tilde{x})\, P_n^\nu(\tilde{x};\varphi_g)\, e^{\left(\varphi_g-\frac{\pi}{2}\right)\tilde{x}},$$

(2.6 a)

Where $\tilde{x} = \frac{mc}{\hbar}x$. Here, the following notations are introduced $\nu = \frac{1}{2}+\sqrt{\frac{1}{4}+\left(\frac{mc^2}{\hbar\omega}\right)^2}$, $\varphi_g = \arccos\frac{g}{mc\omega}$, $0<\varphi_g<\pi$, and the normalization constants are equal to

$$c_n^g = (2\sin\varphi_g)^\nu\sqrt{\frac{mch!}{2\pi\hbar\Gamma(h+2\nu)}}.$$

The explicit form of the wave functions in the momentum representation we find easily with the help of integral transformation with the kernel (2.2):

$$\tilde{\psi}_n^g(p) = c_n^{g\prime} t^\nu \exp(ite^{i\varphi_g})\, L_n^{2\nu-1}(2t\sin\varphi_g),$$

$$c_n^{g\prime} = i^n\sqrt{2\pi}\,c_n^g \exp\left(i\,(n+\nu)\left(\varphi_g-\frac{\pi}{2}\right)\right)$$

(2.6 b)

where $t = c\,(p_0+p)/\hbar\omega$ and $L_n^\alpha(x)$ are the generalized laguerre polynomials. Let us emphasize that the functions (2.6) form the basis for the positive discrete series $D^+(\nu)$ of the irreducible unitary representation of the dynamical symmetry group SU (1.1). The eigenvalue ν of the Casimir operator of this droup does not depend on the force g. Therefore, the wave functions (2.6), corresponding to two different values of the force are connected between themselves by the unitary transformation [11-13], i.e.

$$\psi_n^{g_2} = S_{g_2}S_{g_1}^{+}\psi_n^{g_1},\quad S_g = e^{i\theta_g X_2}, \qquad (2.7)$$

where

$$\theta_g = \frac{1}{2}\ln\frac{mc\omega-g}{mc\omega+g} = \ln tg\frac{\varphi_g}{2}.$$

Using the limiting formulas [13]

$$\lim_{\nu\to\infty}\nu^{-n/2}P_n^\nu\left(x\sqrt{\nu};\arccos\frac{y}{\sqrt{\nu}}\right) = \frac{1}{n!}H_n(x+y),$$

$$\lim_{\nu\to\infty}\nu^{-n/2}L_n^{2\nu-1}(2\nu+2x\sqrt{\nu}) = (-1)^n\frac{1}{n!}H_n(x),$$

where $H_n(x)$ are the Hermite polynomials, it can be easily verifed that in the limit $c\to\infty$ the relativistic wave functions (2.6) coincide with the wave functions of the nonrelativistic linear oscillator in the homogeneous field. In this limit the operator S_g becomes as the particular form of the unitary shift operator $D(\alpha) = \exp(\alpha a^+ - \alpha^* a)$, when , where a^+ and a are usual creation and annihilation operators.

3. Let us now turn to the calculation of the quantum transition probability amplitudes for the relativistic linear oscillator in the homogeneous field, occuring under the action of the suddenly change of the value of the field from $V_{g_1}(x)$ to $V_{g_f}(x)$. Let the oscillator is in the stationary state, described by one of the eigenfunctions $\psi_n^{g_1}$ of the initial Hamiltonian H^{g_1}. After the suddenly change of the field the stationary states of the oscillator will be defined by the wave functions $\psi_m^{g_f}$, which are the eigenfunctions of the new Hamiltonian H^{g_f}. In accordance with the general rules of the quantum mechanics, the trasition probability amplitude S_{mn} of the system from the initial state $\psi_n^{g_1}$ into the final state $\psi_m^{g_f}$ is defined by the coefficients of the expansion of the function $\psi_n^{g_1}$ over the complete and orthonormal set of functions $\{\psi_m^{g_f}\}$:

$$S_{mn} = \int_{-\infty}^{\infty}\psi_n^{g_1}(x)\psi_m^{g_f*}(x)\,dx = \int_{-\infty}^{\infty}\tilde{\psi}_n^{g_1}(p)\tilde{\psi}_m^{g_f*}(p)\,dp/p_0 \qquad (3.1)$$

The relation $\psi_m^{g_f} = S_{g_f}S_{g_1}^{+}\psi_n^{g_1}$ allows us to represent the amplitude (3.1) as a matrix element of the unitary operator $S = S_{g_f}S_{g_1}^{+}$ over the excited by the external field oscillator's basis $\{\psi_n^{g_1}\}$:

$$S_{mn} = (S_{g_f}S_{g_1}^{+})_{mn} = \sum_{k=0}^{\infty}(S_{g_f})_{mk}(S_{g_1}^{+})_{kn}$$

In order to find an explicit form of the matrix element (3.1), it is convenient to carry out the calculation in the momentum representation, where it is a known integral of the laguerre polynomials, a power function and linear exponent of a following form [14]

$$\int_0^{\infty}x^\lambda e^{-ax}L_m^\lambda(bx)L_n^\lambda(cx)\,dx =$$

$$= \frac{(\lambda+1)_m(\lambda+1)_n}{m!\,n!\,a^{m+n+\lambda+1}}\Gamma(\lambda+1)\,(a-b)^m(a-c)^n\,{}_2F_1\left(-m,-n;\lambda+1;\frac{bc}{(a-b)\,(a-c)}\right),$$

$$Re\,a>0,\quad Re\,\lambda>-1,$$

where $(\lambda)_n = \Gamma(n+\lambda)/\Gamma(\lambda)$ is Pochhammer's symbol.

As a result, for the required matrix element (3.1) we obtain the formula

$$S_{mn} = (-1)^n \sqrt{\frac{(2\nu)_n}{m!\,n!\,(2\nu)_n}} \frac{th^{m+n}\frac{\theta_f-\theta_i}{2}}{ch^{2\nu}\frac{\theta_f-\theta_i}{2}} M_n\left(m;2\nu,th^2\frac{\theta_f-\theta_i}{2}\right), \tag{3.2}$$

where $\theta_i \equiv \theta_{q_i}$, $\theta_f \equiv \theta_{q_f}$ and

$$M_n(m;\beta,c) = (\beta)_n\, {}_2F_1\left(-n,-m;\beta,1-\frac{1}{c}\right)$$

are the Meixner polynomials of a discrete variable [15].

Let us now investigate a nonrelativistic limit of the quantity S_{mn}. For this aim we use the ralation [15]

$$\lim_{\beta\to\infty}\left(-\frac{a}{\beta}\right)^n M_n\left(m;\beta,\frac{a}{\beta}\right) = $$
$$= n!\,L_n^{m-n}(a) = (-a)^n C_n(m;a)$$

where $C_n(m;a)$ are the Charlier polynomials of discrete variable and note that when $c\to\infty$ (i.e. when $\nu\to\infty$)

$$th\frac{\theta_f-\theta_i}{2} \to \frac{\alpha_f-\alpha_i}{\sqrt{2\nu}},\quad \left(ch^2\frac{\theta_f-\theta_i}{2}\right)^{-\nu} \to e^{-\frac{(\alpha_f-\alpha_i)^2}{2}},$$
$$\alpha_i \equiv \alpha_{q_i},\quad \alpha_f \equiv \alpha_{q_f}.$$

It is easy now to write a final expression:

$$\lim_{c\to\infty} S_{mn} = \frac{(-1)^n}{\sqrt{m!\,n!}}(\alpha_f-\alpha_i)^{m+n} e^{-\frac{(\alpha_f-\alpha_i)^2}{2}} C_n(m;(\alpha_f-\alpha_i)^2), \tag{3.3}$$

which coincides, as it has been expected, with the matrix elements of the shift operator $D(\alpha)$ with respect to the states of the nonrelativistic linear oscillator at $\alpha=\alpha_f-\alpha_i$ (see, for example, [16]). Therefore, the expression (3.3) defines the transition amplitudes of the nonrelativistic linear oscillator under the action of the suddenly changed homogeneous field. It generalizes the result of the corresponding nonrelativistic porblem, stated in [17].

In conclusion we present a generating function for the amplitude (3.2):

$$\sum_{n,m=0}^{\infty} \sqrt{\frac{(2\nu)_n(2\nu)_m}{n!\,m!}}\,\eta^n\xi^m S_{mn} =$$
$$= ch^{-2\nu}\frac{\theta_f-\theta_i}{2}\left[1+(\xi-\eta)\,th\frac{\theta_f-\theta_i}{2}-\xi\eta\right]^{2\nu}, \tag{3.4}$$

in deriving of which the following formula was used [15]

$$\sum_{n=0}^{\infty} M_n(x;\beta,c)\frac{z^n}{n!} =$$
$$= \left(1-\frac{z}{c}\right)^x (1-z)^{-x-\beta},\quad |z| < min\left(1,|c|\right)$$

It is to be emphasized that the right hand side of (3.4) is indeed accurate within constant the transition amplitudes between the coherent states of the relativistic oscillator.

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RELYATİVİSTİK KVANT OSSİLYATORUNUN BİRCİNS SAHƏ İLƏ HƏYƏCANLAŞMASI. I

Qəfil dəyişən bircins sahənin tə'sirilə relyativistik xətti ossilyatorun n-ci stasionar haldan m-ci stasionar hala kvant keçidlərinin S_{mn} ehtimal amplitudları hesablanmışdır. Göstərilmişdir ki, S_{mn} kəmiyyətləri diskret dəyişənli $M_n(m;\beta,c)$ Meyksner çoxhədliləri ilə ifadə olunur. Onlar üçün törədici funksiya alınmışdır. Onların qeyri-relyativistik limiti tədqiq olunmuşdur. Bu limit uyğun qeyri-relyativistik məsələnin ümumi həllini verir.

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ВОЗБУЖДЕНИЕ РЕЛЯТИВИСТСКОГО КВАНТОВОГО ОСЦИЛЛЯТОРА ОДНОРОДНЫМ ПОЛЕМ. I

Вычислены амплитуды вероятности S_{mn} квантовых переходов из n-го стационарного состояния в m-ое стационарное состояние релятивистского линейного осциллятора под действием внезапно изменяемого однородного поля. Показано, что величины S_{mn} выражаются через полиномы Мейкснера дискретной переменной $M_n(m;\beta,c)$. Для них получена производящая функция. Исследован нерелятивистский предел, который является общим решением соответствующей нерелятивистской задачи.

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