THE AXIOMATICS OF NONRELATIVISTIC QUANTUM MECHANICS. ONE FREE PARTICLE.

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In this paper general structure of axiomatized theory of fundamental physical theory is proposed. This scheme is applied to the construction of quantum mechanics (q, m) of one free particle. The theory is presented in unitary invariant form. From the classical physics the only group of symmetry (Galilean group) is used essentially. The well known relations for the basic observables are obtained and their uniqueness in the frame work of given axiomatics is proved.

1. Introduction

From the papers of methodology and logic of sciences one well knows that the initial system of axioms cannot be estimated inside of axiomatized theory. Its justification is realized with the help of reference to some methodological premises as a generalization of papers an axiomatization of Physics and first of all papers on axiomatics approach to (QFT) [1]. We assume that axiomatized Physical theory contains the next concepts and principles:

1) system of axiom is implicit definition of theoretical objects (TO);
2) the space of states of (TO) has to be presented;
3) the set of initial observable of (TO) has to be pointed out, we denote it by O;
4) the principle of invariance respectively group of symmetry has to be formulated;
5) the axiom of completeness have to be formulated. One needs to explicate the sense in what set of observables pointed in 2) is sufficient for the description of results of possible measurements.;
6) some more special axioms.

In the given paper the nonrelativistic quantum mechanics (q, m) is considered that implies the application of Galilean group G as group of symmetry. In (q, m) the central extension of group G is used. This extension corresponds to some real parameter μ. In this paper the proper subgroup of this group is used rather often. We denote it by G^0_μ.

$$G^0_\mu \subset G_\mu; g \in G^0_\mu; g = g(\theta, \tau, \vec{V}, \vec{a}, r) \quad .$$

where:

$\theta \in R^1$ is parameter of central extension. Denote by M the generator of corresponding one-parameter subgroup.

$\tau \in R^1$ is parameter of time translation. Denote by H corresponding generator.

$\vec{V} \in R^3$ is parameters of proper Galilean transformation (boost). Denote by $\vec{K}$ corresponding generators.

$\vec{a} \in R^3$ are parameter of space translation. Denote by $\vec{P}$ corresponding generators.

$r \in O(3)$ - parameters of rotation. Denote by $\vec{J}$ corresponding generators.

In order to facilitate the verification of following propositions we present here the law of group production and commutation relation for algebra of group G^0_μ [2].

$$\begin{aligned} &g_1 \circ g_2 = g(\theta_1 + \theta_2 + \xi_{12}, \tau_1 + \tau_2, \vec{V}_1 + r_1\vec{V}_2, \vec{a}_1 + r_1\vec{a}_2 + \tau_2\vec{V}_1, r_1 r_2) \\ &\xi_{12} = \mu\left(\frac{1}{2}V_1^2\tau_2 + (\vec{V}_1, r_1\vec{a}_2)\right) \end{aligned} \qquad (1)$$

$$\left.\begin{aligned} &[J_\alpha, J_\beta] = i\varepsilon_{\alpha\beta\gamma}J_\gamma; [J_\alpha, P_\beta] = i\varepsilon_{\alpha\beta\gamma}P_\gamma; [P_\alpha, P_\beta] = 0 \\ &[J_\alpha, K_\beta] = i\varepsilon_{\alpha\beta\gamma}K_\gamma; [K_\alpha, K_\beta] = 0; [K_\alpha, P_\beta] = i\delta_{\alpha\beta}M \\ &[J_\alpha, M] = 0; [P_\alpha, M] = 0; [K_\alpha, M] = 0; [H_\alpha, M] = 0 \\ &[H, \vec{P}] = 0; [H, \vec{J}] = 0; [H, \vec{K}] = -i\vec{P} \end{aligned}\right\} \qquad (2)$$

2. Quantum mechanics of one free particle.

In this paragraph we introduce the complete family of axioms which are necessary for description of nonrelativistic quantum mechanical particle.

Axiom I (state space): The state space of (q, m) particle is separable Gilbert space ($\mathcal{H}$).

Axiom II (set of observables): The set of observables is some von-Neumann algebra $\mathcal{M}$, $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ which contains 6 self-adjoint operators $\vec{Q}$ and $\vec{\Sigma}$, satisfying the following bilinear conditions

$$\begin{aligned} &[Q_\alpha, Q_\beta] = 0; [\Sigma_\alpha, Q_\beta] = 0 \\ &[\Sigma_\alpha, \Sigma_\beta] = i\varepsilon_{\alpha\beta\gamma}\Sigma_\gamma; \vec{\Sigma}^2 = \vec{\sigma}^2 1; \vec{\sigma}^2 = \sigma(\sigma+1) \end{aligned} \quad (3)$$

where σ is half-integer number.

Axiom II' (completeness): $\mathcal{W}^*$ algebra generated by operators $\vec{Q}$ and Σ_3 is maximal commutative algebra in $\mathcal{H}$. For the formulating of the axiom of invariance we need one notation. Consider the following subset of elements of group G_μ^0

$$\mathcal{E}_3 = \{g \in G_\mu^0; g(0,0,\vec{0},\vec{a},r); \vec{a} \in R^3, r \in O(3)\} \quad (4)$$

Obvious $\mathcal{E}_3$ is subgroup in G_μ^0 it is called Euclidean group.

Axiom III$_1$ (invariance respectively Euclidean group): There exists unitary representation of group $\mathcal{E}_3$ $U(\cdot) \in Rp_p(\mathcal{E}_3, \mathcal{H})$ such that under transformation from one system reference to another system connected with the first one by transformation $\varepsilon \in \mathcal{E}_3$, observables are transformed respectively law

$$\begin{aligned} U(\vec{a})V(\vec{b})U(\vec{a})^* &= V(\vec{b})e^{i\vec{a}\vec{b}} \\ U(r)V(\vec{b})U(r)^* &= V(r\vec{b}) \\ U(\varepsilon)\vec{\Sigma}U(\varepsilon)^* &= r\vec{\Sigma} \end{aligned}$$

where $V(\vec{b}) = e^{i\vec{Q}\vec{b}}$

In the following theorem I the general form of these operators is obtained.

Theorem I: There exist two Hilbert spaces $\mathcal{F}$ and $\mathcal{L}$ such that state space $\mathcal{H}$, observables $\vec{Q}, \vec{P}, \vec{\Sigma}$ are unitary equivalent to the following set of operators

$$(\mathcal{H}, \vec{Q}, \vec{P}, \vec{\Sigma}) \cong (\mathcal{L} \otimes \mathcal{F}, \vec{p}_0 \otimes 1_{\mathcal{F}}, \vec{q}_0 \otimes 1_{\mathcal{F}}, 1_{\mathcal{L}} \otimes \vec{S}) \quad (5)$$

where $(\mathcal{L}, \vec{p}_0, \vec{q}_0)$ are unitary equivalent to Schrödinger representation of CCR, and operators $\vec{S}$ realize in Hilbert space $\mathcal{F}$ irreducible representation of algebra of group $SU(2)$, which corresponds to spin σ.

The proof of theorem I is based on application of von-Neumann's theorem about uniqueness of representation of CCR with finite degrees of freedom. [3]. One also needs to use axiom II' and the method of construction of representation of $SU(2)$, proposed in [4].

Now we introduce some algebras and relations between them which will be used in this paper and for consideration of many particle case. Consider the following von-Neumann algebras

$$\left.\begin{aligned} \mathcal{M} &= \mathcal{W}^*\{V(\vec{b}), \vec{\Sigma}\}_{b \in R^3} \\ \mathcal{M}_0 &= \mathcal{W}^*\{V(\vec{b})\}_{b \in R^3} \\ \mathcal{K} &= \mathcal{W}^*\{V(\vec{b}), \Sigma_3\}_{b \in R^3} \end{aligned}\right\} \quad (6)$$

These algebras satisfy some properties that will be used to prove the theorems of uniqueness. We present these properties in the form of the theorem II.

Theorem II: The algebras defined by formulas (6) satisfy the following relations:

i) $\mathcal{K}$ is maximal commutative algebra;

ii) $\mathcal{M}' - \mathcal{M}_0$;

iii) $\mathcal{W}^*(\{V(\vec{b}), U(\vec{a}), \vec{\Sigma}\}_{\vec{b} \in R^3, \vec{a} \in R^3}) = B(\mathcal{H})$

The proof of this theorem is based on general properties of von-Neumann's algebras and the next result $\vec{q}_0$ generate in $\mathcal{L}$-maximal commutative algebra [5].

To prove the theorem of uniqueness for operators $\vec{P}$ (theorem III) we extend the group of symmetry with the help one new transformation i_t, which is transformation of time reflection. It is connected with another elements of group G_μ^0 by the following relations

$$i^2 = I \in G_\mu^0; igi^{-1} = \hat{g} = g(\theta, -\tau, -\vec{V}, ar) \quad (7)$$

Using relation (7) and analogy with classical mechanics we assume the next axiom

Axiom III$_2$ (invariance respectively time reflection): $g \in G_\mu^0, U(\cdot) \in R_p(G_\mu^0, \wp)$, I- is representation i_t by antiunitary in $\mathcal{H}$. Assume, that for representation of G_μ and for algebra of initial observables the following relations take place.

$$\left.\begin{aligned} &IU(\varepsilon)I^{-1} = U(\varepsilon) \\ &IU((\vec{V}))I^{-1} = U((-\vec{V})), |\vec{V}| = g(0,0,\vec{V},0,1) \\ &IU((\tau))I^{-1} = U((-\tau)), |\tau| = g(0,\tau,\vec{0},\vec{0},1) \\ &I^2 = 1_{\mathcal{H}} \end{aligned}\right\} \quad (8)$$

$$\left.\begin{aligned} IV(\vec{b})I^{-1} &= V(-\vec{b}) \\ I\vec{\Sigma}I^{-1} &= -\vec{\Sigma} \end{aligned}\right\} \quad (9)$$

We note that (8) is representation of composition law (7). And relation (9 for spin is chosen analogous to the same relation for generators of angular momentum. The next theorem of uniqueness is true.

Theorem III: Let $(U(\varepsilon), I)$ and $('U(\varepsilon), 'I)$ be two representations of group $\mathcal{E} \otimes i_t$ that provide the realization of axiom III$_1$ and III$_2$ for the same set of initial observables

$V(\vec{b})$ and $\vec{\Sigma}$. Assume also that these two representations are connected by the next relations

$$'I = e^{i\alpha} I \quad \alpha \in R^1 \qquad (10')$$

$$D(\vec{P}) \cap D('\vec{P}) = core(\vec{P}) \qquad (10'')$$

Then $\vec{P} = '\vec{P}$ and $U(\vec{a}) = 'U(\vec{a})$.

For the proof of this theorem is used relation $ITI^{-1} = T^* \quad \forall T \in \mathcal{M}_0$ and (ii) of theorem II.

It is clear that the same result is true for operators of rotation. But in order to avoid the second use of unsatisfactory condition (10'). We present another formulation of this theorem.

Theorem IV: Let $U(\varepsilon)$ and $'U(\varepsilon)$ be two different representation of group $\mathcal{E}_3$ and for these representations axiom III_1 is performed for the same observables $\vec{a}$ and $\vec{\Sigma}$. Assume also that the next relations between $U(\varepsilon)$ and $'U(\varepsilon)$ are performed.

$$U(\vec{a}) = 'U(\vec{a})$$
$$D(\vec{J}) \cap D('\vec{J}) = core(\vec{J})$$

Then i) $U(r) = 'U(r)$

ii) and operators of angular momentum have the following form :

$$J_\alpha = \varepsilon_{\alpha\beta\gamma} Q_\beta P_\gamma + \Sigma_\alpha = L_\alpha + \Sigma_\alpha$$

The proof of i) is based of verification of the following relation $U(r)\,'U(r)^* \in \{V(\vec{b}), V(\vec{a}), \vec{\Sigma}\}'$ and application to this relation (iii) of theorem II.

i) Asserts that this well known expression for operator of angular momentum is uniquely possible in the frame work of given axiomatics that is direct derivation of (ii).

In order to complete the presentation of family of axioms invariance. We claim the axiom of invariance respectively boost transformations. From classical mechanics one knows that if $R^*(\vec{x}, t)$ is space of events then the boost transformation acts there as follows

$$(\vec{v}) : (\vec{x}, 0) = (\vec{x}, 0) \qquad (11)$$

Axiom III_4 is quantum mechanical generalization of this relation.

Axiom III_4 (invariance respectively boost transformation): There exists such unitary representation of group G_μ^0, that boost transformation connected with initial observables as follows.

$$U((\vec{v}))V(\vec{b})U^*((\vec{v})) = V(\vec{b})$$
$$U((\vec{v}))\vec{\Sigma}U((\vec{v}))^* = \vec{\Sigma} \qquad (12)$$

For the completeness of physical interpretation we need new axiom.

Axiom IV (spectrality): Generator of phase transformation has the following form $M = m1_H, \quad m > 0$ ....

Now we are able to prove the theorem on general form of boost transformation.

Theorem V: Let axioms (I-IV) be performed, then the generator of boost transformation has the following form

$$\vec{K} = m\vec{Q} \qquad (13)$$

Proof. Consider operator $\chi(\vec{v}) = V^*(m\vec{v})U((\vec{v}))$. Using composition low (1) and relations (5) one can verify $[\chi(\vec{v}), U(\vec{a})] = 0$ Relations (12) also imply that $[\chi(\vec{v}), V(\vec{b})] = 0$, $[\chi(\vec{v}), \vec{\Sigma}] = 0$. Then applications (iii) of theorem II and invariance respectively rotation give $\chi(\vec{v}) = a(v^2)1_H$. From this relation by differentiation we get (13).

Consider subset of Galilean group

$$G_\mu^{st} = \{g \in G_\mu^0 : g = g(\theta, 0, \vec{V}, \vec{a}, r), \theta \in R^1, \vec{V} \in R^3, \vec{a} \in R^3, r \in O(3)\}$$

Obvious that this subset is subgroup in G_μ^0 we call this set stationary subgroup.

Let us summarize our result. The theory designed by axioms (I-IV) determines initial observables up to unitary equivalence. Generators of stationary subgroup are also determined almost uniquely. We call this representation of G_μ^0 admissible one. For the description of complete symmetry we have to determine the invariance respectively time translation. It means that we need to define self-adjoint operator H such, this operator and other operators which provide admissible representation of Lie algebra of G_μ^0. Satisfy commutation relation (2). This operator H is called admissible Hamiltonian.

Theorem VI: Let all axioms (I-IV) be performed and exist linear dense set $D_1 \subset D(\vec{Q}) \cap D(\vec{P}), \vec{Q}, \vec{P}: \ D_1 \to D_1$ and D_1 is set of essential self-adjointeness of generator of group G_μ^{st}. Then every admissible Hamiltonian has form

$$H = \frac{p^2}{2m} + E1_H, \quad E \in R^1 \qquad (15)$$

That the same idea is used for the proof, if H is admissible Hamiltonian, then $\left(H-\frac{p^2}{2m}\right)\in\{U(\vec{a}),V(\vec{b}),\vec{\Sigma}\}'$ then everything is obvious.

From the theory of unitary representation of Galilean group [6] one knows, that representations are irreducible. If and only if operators $M,(J-h)^2,H-\frac{p^2}{2m}$ are proportional to identical operators. Namely these properties are performed in our axiomatics. So the next theorem is true.

Theorem VII: In the framework of proposed axiomatics the properties of symmetry of one quantum mechanical particle are described by unitary irreducible representation of centrally extended Galilean group corresponding to any positive mass m, half-integer values of spin σ, and constant internal energy E.

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QEYRİ-RELYATİVİSTİK KVANT MEXANİKASININ AKSİOMATİKASI. BİR SƏRBƏST ZƏRRƏCİK.

Bu məqalədə aksiomatik fundamental fiziki nəzəriyyənin əsasları verilmişdir. Bu ümumi sxem bir sərbəst zərrəcikli kvant mexanikasının yaradılmasına tətbiq edilmişdir. Nəzəriyyə unitar-invariant şəkildə ifadə olunmuşdur. Nəzəriyyə yaradılan zaman klassik fizikadan yalnız simmetriya qrupu (Qaliley qrupu) istifadə olunmuşdur. Bütün əsas müşahidə olunan fiziki kəmiyyətlərin ifadələri alınmışdır və təklif olunan aksiomatika çərçivəsində onların vahidliyi isbat olunmuşdur.

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АКСИОМАТИКА НЕРЕЛЯТИВИСТСКОЙ КВАНТОВОЙ МЕХАНИКИ. ОДНА СВОБОДНАЯ ЧАСТИЦА.

В данной статье предложена структура аксиоматизированной фундаментальной физической теории. Эта общая схема применена к построению квантовой механики одной свободной частицы. Теория сформулирована в унитарно-инвариантной форме. При построении теории из классической физики существенно использование только группы симметрии (группа Галилея). Получены выражения для всех основных наблюдаемых и доказана их единственность в рамках предложенной аксиоматики.

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