WIGNER FUNCTION OF RELATIVISTIC LINEAR OSCILLATOR

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The obvious form of Wigner function was found for relativistic linear oscillator. It was shown that it has correct nonrelativistic limit. The density of probability distribution and average value of energy were calculated by means of Wigner function.

1. In the most annexes of quantum mechanics the Wigner function W(p,q,t) has found large applying [1-5]. It is a quantum analogy of the classic function of a statistical distribution on the phase space $\rho(p,q)$. The relationship $\lim_{h\to 0} W(p,q,t)=\rho(p,q)$ has a place and of course one can find quantum corrections to the classical results by means of Wigner representation of quantum mechanics. The Wigner function can be obtained from the wave function of system in a coordinate q- or momentum p-representation by means of

$$W(p,q,t)=\frac{1}{2\pi h}\int_{-\infty}^{\infty}\psi^*(q+\xi/2,t)\psi(q-\xi/2,t)e^{\frac{ip\xi}{h}}d\xi \quad (1.1a)$$

$$W(p,q,t)=\frac{1}{2\pi h}\int_{-\infty}^{\infty}\psi^*(p+\eta/2,t)\psi(p-\eta/2,t)e^{\frac{iq\eta}{h}}d\eta \quad (1.1b)$$

The function (1.1) is real, but not satifying the condition of positivity and therefore it can't be considered as a density of probability in the phase space of the coordinates and momentums. However, it has a property, that

$$\int_{-\infty}^{\infty}W(p,q,t)dp=W(q,t)\ ,\quad \int_{-\infty}^{\infty}W(p,q,t)dq=W(p,t) \quad (1.2)$$

where $W(q,t)=|\psi(q,t)|^2$ and $W(p,t)=|\psi(p,t)|^2$ - densities of distribution of probabilities along coordinates and momentums. One can calculate the average value of any physical quantity f(p,q) by means of Wigner function by formula

$$\bar{f}=\int f(p,q)W(p,q,t)dpdq \quad (1.3)$$

For a some of nonrelativistic quantum mechanical systems the obvious form of Wigner function was found in the works [2,7,8,10,11].

The purpose of the present work is to find the obvious form of the Wigner function for relativistic model of the linear oscillator examinated detailly in the works [12,13].

2. In the relativistic configurated x-presentation the relativistic model of the linear oscillator is described by finite-difference operator

$$\hat{H}(x)=mc^2\cdot \mathrm{ch}\, i\lambda\partial_x+\frac{m\omega^2}{2}x(x+i\lambda)e^{i\lambda\partial_x} \quad (2.1)$$

where $\lambda=h/mc$ - Kompton wave length. The eigenfunctions of hamiltonian H(x) satisfying the ortogonality condition

$$\int_{-\infty}^{\infty}\psi_n^*(x)\psi_m(x)dx=\delta_{nm} \quad (2.2)$$

has a form

$$\psi_n(x)=C_n\cdot\alpha^{ix/\lambda}\Gamma(\nu+ix/\lambda)P_n^{\nu}(x/\lambda;\pi/2) \quad (2.3)$$

The normalization coefficients $C_n=2^{\nu}[n!/2\pi\lambda\Gamma(n+2\lambda)]^{1/2}$, where $2\nu=1+\sqrt{1+4\alpha^2}$, $\alpha=mc^2/h\omega$, and $P_n^{\nu}(x;\varphi)$ - Meixner-Pollachzek polynomials. The eigenvalues of the hamiltonian H(x) with a according wave functions $\psi_n(x)$ are $E_n=h\omega(n+\nu)$, n=0, 1, 2, ...

The transition to the momentum representation considering as a one-dimensional Lobachevsky space realized on the hyperbola $p_0^2-p^2=m^2c^2$, $p_0>0$ in the surface (p_0,p)

$$\psi_n(p)=\frac{1}{\sqrt{2\pi h}}\int_{-\infty}^{\infty}\xi^*(p,x)\psi_n(x)dx \quad (2.4)$$

is realized by means of the decomposition along relativistic surface waves $\xi(p,x)=((p_0+p)/mc)^{ix/\lambda}=e^{i\chi x/\lambda}$, where $\chi=\ln((p_0+p)/mc)$ - rapidity. The wave functions are expressed by Laguerre polynomials

$$\psi_n(P)=C'_n t^{\nu}L_n^{2\nu-1}(2t)e^{-t},\quad t=C(P_0+P)/h\omega=\alpha e^{\chi},\quad(2.5)$$

$$C'_n=i^n\sqrt{2\pi\lambda/mc}\,C_n$$

and satisfying the ortogonality condition

$$\int\psi_n^*(P)\psi_m(P)d\Omega_p=mc\int_{-\infty}^{\infty}\psi_n^*(P)\psi_m(P)d\chi=\delta_{nm},\quad (2.6)$$

$$d\Omega_p=mcdP/P_0=mcd\chi$$

3. Let's determine the Wigner function of the stationary states in the considering case in analogy with (1.1) in following

$$W_n(\chi,x)=\frac{1}{2\pi h}\int_{-\infty}^{\infty}\psi_n^*(x+\xi/2)\psi_n(x-\xi/2)e^{i\chi\xi/\lambda}d\xi= \quad (3.1a)$$

$$=\frac{1}{2\pi h}\int_{-\infty}^{\infty}\psi_n^*(\chi+\xi/2)\psi_n(\chi-\xi/2)e^{-ix\eta/\lambda}d\eta \qquad (3.1б)$$

The Wigner function determined in the momentum representation in the form are connected with that the quantities X and χ are canonical-conjugate variables in the meaning of Fourier transformations (2.4) [14].

Substuting in (3.1b) $\psi_n(\chi)=\psi_{n}(P)$ from (2.5) we obtain:

$$W_n(\chi,x)=\frac{n!}{(2\nu)_n}L_n^{2\nu-1}(2te^{\beta/2\pi})L_n^{2\nu-1}(2te^{-\frac{\beta}{2}})W_0(\chi,x)=$$

$$\frac{2n!(2t)^{2\nu}}{\pi h\Gamma(n+2\nu)}\sum_{k,s=0}^{n}\begin{pmatrix}n+2\nu-1\\ n-k\end{pmatrix}\begin{pmatrix}n+2\nu-1\\ n-s\end{pmatrix}\frac{(-2t)^{k+s}}{k!s!}K_{2ix/\lambda,s,k}(2t) \qquad (3.2)$$

Here for the Wigner function $W_0(\chi,X)$ of the ground state of the relativistic linear oscillator (2.1) we have expression

$$W_0(\chi,X)=\frac{2(2t)^{2\nu}}{\pi h\Gamma(2\nu)}K_{2ix/\lambda}(2t) \qquad (3.3)$$

and $K_\nu(z)$ - Macdonald function. $W_0(\chi,X)$ has a correct nonrelativistic limit:

$$\lim_{c\to\infty}W_0(\chi,x)=\frac{1}{\pi h}\exp\left\{-\frac{2}{h\omega}\left(\frac{P^2}{2m}+\frac{m\omega^2x^2}{2}\right)\right\} \qquad (3.4)$$

Equilibrium Wigner Function is determined by equality [2,7]

$$W(\chi,x)=Z^{-1}(\beta)\sum_{n=0}^{\infty}e^{-\beta E_n}W_n(\chi,x) \quad , \quad \beta=1/kT \qquad (3.5)$$

where T - temperature, k- Boltsmann constant, and normalization multiplier

$$Z(\beta)=\sum_{n=0}^{\infty}e^{-\beta E_n}=\frac{z^\nu}{1-z} \quad , \quad z=e^{-\beta h\omega} \qquad (3.6)$$

is a statistical sum of the relativistic linear oscillator.

Substuting the integral representation (3.1b) in (3.5) and changing an order of the sum and integration one can write (3.5) in the following:

$$W(\chi,x)=\frac{1-z}{\pi h}(2t)^{2\nu}\int_{-\infty}^{\infty}d\eta\cdot\exp(-2t\,ch\eta+2ix\eta/\lambda)\cdot$$

$$\cdot\sum_{n=0}^{\infty}\frac{n!z^n}{\Gamma(n+2\nu)}L_n^{2\nu-1}(2te^{\eta})L_n^{2\nu-1}(2te^{-\eta}) \qquad (3.7)$$

Then taking into account the following bilinear generating function for a Laguerre polynomials [15]

$$\sum_{n=0}^{\infty}\frac{n!z^n}{\Gamma(n+2+1)}L_n^{\alpha}(x)L_n^{\alpha}(y)=(1-z)^{-1}\exp\left[-\frac{z(x+y)}{1-z}\right]\cdot$$

$$\cdot(xyz)^{-\alpha/2}\cdot I_\alpha\left(\frac{2\sqrt{xyz}}{1-z}\right)$$

we obtain after integration the obvious expression for the equilibrium Wigner function of relativistic linear oscillator:

$$W(\chi,x)=\frac{4t}{\pi h}z^{-\nu+\frac{1}{2}}\cdot I_{2\nu-1}(2at)K_{2ix/\lambda}(2bt),$$
$$a=1/sh\frac{\beta h\omega}{2} \quad , \quad b=cth\frac{\beta h\omega}{2} \qquad (3.8)$$

Let's calculate now the density of the momentum distribution (1.2) and average value of the energy E of relativistic linear oscillator by means of Wigner function (3.8):

1.According to (1.2) under the circumstances the momentum distribution is

$$W(\chi)=\int_{-\infty}^{\infty}W(\chi,x)dx=\frac{2z^{-\nu+\frac{1}{2}}}{mc}t\cdot I_{2\nu-1}(2at)e^{-2bt} \qquad (3.9)$$

2. For calculating the average value of energy at first we"ll find Weyl ordering form $H(\chi,X)$ according to Hamilton operator $\hat{H}(x)$ (2.1) - Hamilton function

$$H(\chi,x)=\int e^{i\chi\xi/\lambda}<x-\xi/2|\hat{H}(x)|x+\xi/2>d\xi$$

where $|x>$ - eigenvector of the operator $\hat{x}=x$. We find from here that

$$H(\chi,x)=mc^2ch\chi+\frac{m\omega^2}{2}\left(x^2+\frac{\lambda^2}{4}\right)\cdot e^{-\chi} \qquad (3.10)$$

and consiquently,

$$\overline{E}=mc\int_{-\infty}^{\infty}H(\chi,x)W(\chi,x)dxd\chi$$

After calculating of integration, we find the "relativistic" Plank formula

$$\overline{E}=h\omega\nu+\frac{h\omega}{e^{\frac{h\omega}{kT}}-1} \qquad (3.11)$$

The same result can be easily obtained by means of the statistical sum (3.6)

$$\overline{E}=-\frac{\partial}{\partial\beta}\ln Z(\beta)=h\omega\left(\nu-\frac{1}{2}\right)+\frac{1}{2}h\omega\,cth\frac{\beta h\omega}{2} \qquad (3.12)$$

Here the first term in right-hand side of (3.12) is connected with rest energy of the relativistic oscillator.

Let's note in the end that it is not difficult to write difference equation, which is satisfied by Wigner function.

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RELYATİVİSTİK XƏTTİ OSSİLYATORUN VİONER FUNKSİYASI

Relyavistik xətti ossilyator üçün Viqner funksiyasının aşkar şəkli tapılmışdır. Göstərilmişdir ki, o düzgün qeyri-relyavistik limitə malikdir. Viqner funksiyasının köməyi ilə ehtimalların paylanma sıxlığı və enerjinin orta qiyməti hesablanmışdır.

Ш.М.НАГИЕВ, Э.И.ДЖАФАРОВ

ФУНКЦИЯ ВИГНЕРА РЕЛЯТИВИСТСКОГО ЛИНЕЙНОГО ОСЦИЛЛЯТОРА

Найден явный вид функции Вигнера для релятивистского линейного осциллятора. Показано, что она обладает правильным нерелятивистским переделом. С помощью функции Вигнера вычислены плотность распределения вероятностей и среднее значение энергии.

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