ON PHASE-SPACE DISTRIBUTION FUNCTIONS FOR THE NONLOCAL RELATIVISTIC OSCILLATOR

Sh.M. NAGIYEV, E.I. JAFAROV

Institute of Physics, Academy of Sciences of Azerbaijan,
370143, Baku, H. Javid av., 33

The phase-space representation of the finite-difference relativistic quantum mechanics is considered. The Wigner and standard-ordered distribution functions for the nonlocal relativistic model of the linear oscillator and their nonrelativistic limits are calculated. Averages in terms of the Wigner distribution functions of the coordinate, momentum and energy (Plank's formula) are found.

1. The representation of quantum mechanics in (q,p) phase-space (PS) in terms of distribution functions (d.f.) [1,2] widely used in all areas of physics including statistical physics [3], quantum optics [4], collision theory [5]. The PS representation offers a framework in which classical languages are allowed. It requires dealing only with c-number equations and not with operators. The PS d.f. allow one to express quantum mechanical averages in a form which is very similar to that for classical averages. Thus, the average of an arbitrary operator $\hat{f}(\hat{q},\hat{p})$ that corresponds to a certain quantum mechanical observable can be calculated using the d.f. $F(q,p,t)$ as

$$\bar{f} = \int f(q,p)\,F(q,p,t)\,dp\,dq \tag{1.1}$$

The scalar function $f(q,p)$ can be derived from the operator $\hat{f}(\hat{q},\hat{p})$ by a well defined correspondence rule.

It is well known that there is no unique way of defining the quantum PS d.f. $F(q,p,t)$ due to the noncomutability of quantum mechanical coordinate and momentum operators $\hat{q}$ and $\hat{p}$.

A general class of the quantum PS d.f. for pure states can be defined by the following equation [2,6]

$$F^{f}(q,p,t)=\frac{1}{(2\pi\hbar)^2}\int_{-\infty}^{\infty} d\xi\, d\eta\, dq'\cdot\psi^*\left(q'-\frac{\eta}{2},t\right)\cdot\psi\left(q'+\frac{\eta}{2},t\right)\cdot f(\xi,\eta)\, e^{\frac{i\xi(q'-q)}{\hbar}}\, e^{-i\eta p/\hbar} \tag{1.2}$$

where $\psi(q,t)$ is a wave function in a coordinate representation.

A difference choice of the function $f(\xi,\eta)$ corresponds to a different PS d.f.. We shall here consider only two choices.

If $f(\xi,\eta)=1$ we obtain the Wigner d.f.[7]

$$F^{W}(q,p,t)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} d\eta\cdot\psi^*\left(q'-\frac{\eta}{2},t\right)\cdot\psi\left(q'+\frac{\eta}{2},t\right)e^{-i\eta p/\hbar} \tag{1.3}$$

If $f(\xi,\eta)=e^{-i\xi\eta/2\hbar}$ we obtain the standard-ordered d.f. [8]

$$F^{S}(q,p,t) = \frac{1}{\sqrt{2\pi\hbar}}\,\psi^*(q,t)\cdot\tilde{\psi}(p,t)\cdot e^{ipq/\hbar} \tag{1.4}$$

where $\tilde{\psi}(p,t)$ is the wave function in momentum space

$$\tilde{\psi}(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int dq\cdot\psi(q,t)\cdot e^{-ipq/\hbar} \tag{1.5}$$

The explicit form of the PS d.f. for a number of nonrelativistic quantum mechanical problems were found in [2,9,10]. Such expressions were obtained also for the Wigner d.f. one of the relativistic oscillator model [1].

The purpose of the present paper is to find the explicit expressions for the PS d.f. for another relativistic oscillator, considered in [12].

2. In the configurational x-realization the relativistic model of the linear oscillator under discussion is described by the equation [12].

$$(H_0 - E)\psi(x) + \int_{-\infty}^{\infty} V(x,x')\psi(x')\,dx' = 0 \tag{2.1}$$

where the free Hamiltonian has the form

$$H_0 = mc^2 \operatorname{ch} i\lambda\partial_x,\quad \partial_x = d/dx \tag{2.2}$$

and the oscillator quasipotential is nonlocal

$$V(x,x') = \frac{\hbar\omega}{\lambda^3 c}\,\frac{xx'(x-x')}{\operatorname{sh}(\pi(x-x')/\lambda)} \tag{2.3}$$

The wave functions can be presented as

$$\psi_n(x) = \frac{(-1)^n}{\sqrt{2^n - n!}} H_n\left(\sqrt{2}\Lambda sh\frac{i\lambda}{2}\partial_x\right)\psi_0(x),$$
$$\psi_0(x) = \sqrt{\frac{2}{\pi h}} C_0 mce^{\Lambda^2/2} \cdot K_{ix/\lambda}(\Lambda^2/2) \quad (2.4a)$$

Here $H_n(x)$ is the Hermite polynomial, $K_{ix}(z)$ is the Makdonald function and $\Lambda = \sqrt{2mc^2/h\omega}$. In the momentum $p=mc\ sh\chi$- realization we have

$$\Phi_n(\chi) = c_n e^{-K_p^2/2mh\omega} \cdot H_n(K_p/\sqrt{mh\omega}),$$
$$K_p = 2mcsh\frac{\chi}{2},\quad c_n = c_0/\sqrt{2^n \cdot n!},\quad c_0 = (1/\pi m\omega h)^{1/4} \quad (2.4b)$$

The energy spectrum is

$$E_n = mc^2 + h\omega\left(n + \frac{1}{2}\right),\quad n = 0,1,2,\ldots \quad (2.5)$$

3. We define Wigner d.f. for the stationary states of the relativistic linear oscillator (2.3) as [11]

$$F_n^W(\chi,x) = \frac{1}{2\pi\lambda}\int \psi_n^*\left(x+\frac{\xi}{2}\right)\cdot\psi_n\left(x-\frac{\xi}{2}\right)e^{i\chi\xi/\lambda}d\xi =$$
$$= \frac{1}{2\pi\lambda}\int \Phi_n^*\left(\chi+\frac{\eta}{2}\right)\cdot\Phi_n\left(\chi-\frac{\eta}{2}\right)e^{-ix\eta/\lambda}d\eta \quad (3.1)$$

Taking into account the formulas (2.4) we can obtain

$$F_n^W(\chi,x) = \frac{1}{2^n \cdot n!} H_n\left(ae^{\frac{i}{4}\lambda\partial_x} - be^{-\frac{i}{4}\lambda\partial_x}\right)\cdot H_n\left(ae^{-\frac{i}{4}\lambda\partial_x} - be^{\frac{i}{4}\lambda\partial_x}\right)\cdot F_0^W(\chi,x) \quad (3.2a)$$

where Wigner d.f. for the ground state is equal to

$$F_0^W(\chi,x) = \frac{2c_0^2}{\pi\lambda}e^{\Lambda^2}\cdot K_{2ix/\lambda}(\Lambda^2 ch\chi) \quad (3.3)$$

and $a = \frac{\Lambda}{\sqrt{2}}e^{\chi/2}$, $b = \frac{\Lambda}{\sqrt{2}}e^{-\chi/2}$. In the nonrelativistic limit we have a correct result:

$$F_0^W(\chi,x) \underset{c\to\infty}{\to} \frac{1}{\pi h}e^{-\frac{1}{h\omega}\left(\frac{p^2}{m}+m\omega^2x^2\right)} \quad (3.4)$$

We can also presented $F_n^W(\chi,x)$ as follows:

$$F_n^W(\chi,x) = \frac{2c_0^2(\Lambda^2e^\chi)^n}{\pi\lambda}e^{\Lambda^2}\, n!\sum_{1,1'=0}^{\left[\frac{n}{2}\right]}\frac{(-e^{-\chi/2\Lambda^2})^{1+1'}}{1!\,1'!\,(n-21)!\,(n-21')!}$$
$$\sum_{s=0}^{n-21}\sum_{s'=0}^{n-21'}C_{n-21}^{s}C_{n-21'}^{s'}(-e^{-\chi})^{s+s'}\cdot K_{2ix/\lambda+1-1'+s-s'} \quad (3.2b)$$

Let us consider now the equilibrium Wigner d.f.

$$F^W(\chi,x) = Z^{-1}(\beta)\sum_{n=0}^{\infty}e^{-\beta E_n}F_n^W(\chi,x),\quad \beta = 1/kT \quad (3.5)$$

where the partition function $Z(\beta)$ for the relativistic oscillator (2.3) has the form

$$Z(\beta) = \sum_{n=0}^{\infty}e^{-\beta E_n} = \frac{e^{-\beta mc^2}}{2shf},\quad f = \beta h\omega/2 \quad (3.6)$$

After some calculations we find at the final expression

$$F^W(\chi,x) = \frac{2\Lambda}{\pi h}\sqrt{\frac{2}{\pi}thf}\cdot\exp\left\{\Lambda^2 cthf + \frac{K_p^2}{mh\omega\cdot sh2f}\right\}K_{2ix/\lambda}\left(\Lambda^2 cthf + \frac{K_p^2}{mh\omega}cth2f\right) \quad (3.7)$$

It has a correct nonrelativistic limit. Both the functions (3.2) and (3.7) satisfy normalization condition:

$$\int F_n^W(\chi,x)\ dk_p dx = 1,\quad \int F^W(\chi,x)\,dk_p dx = 1$$

The equilibrium Wigner d.f. (3.7) leads to the Gaussian distribution for the momentum

$$W(\chi) = \int_{-\infty}^{\infty}F^W(\chi,x)\,dx = \frac{1}{\sqrt{\pi\sigma}}e^{-k_p^2/\sigma} \quad (3.8)$$

with the width $\sigma = m\hbar\omega\, cthf$.

From (3.6) we obtain following expression for the average of the energy (Plank's formula)

$$\overline{E} = -\frac{\partial}{\partial\beta} \ln Z(\beta) = mc^2 + \frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{2f} - 1}$$

One can compute the averages of the coordinate and momentum for the relativistic oscillator (2.3) with respect to the Wigner d.f.:

$$\overline{x}_n = \int dx dk_p \cdot x \cdot F_n^W(\chi, x) = 0$$

$$\overline{p}_n = mc \int dx dk_p \cdot shx \cdot F_n^W(\chi, x) = 0$$

Similarly have $\overline{x} = 0$ and $\overline{p} = 0$.

The standard-ordered d.f. for the stationary states or the relativistic oscillator (2.3) can be presented using (1.4) as

$$F_n^S(\chi, x) = \frac{1}{\sqrt{2\pi\hbar}} \psi_n^*(x) \Phi_n(\chi)\, e^{i\chi x/\lambda} \qquad (3.9)$$

We define now in analogy with (3.5) the equilibrium standard-order d.f.

$$F^S(\chi, x) = Z^{-1}(\beta) \sum_{n=0}^{\infty} e^{-\beta E_n} F_n^S(\chi, x) \qquad (3.10)$$

After some transformations we can show that

$$F^S(\chi, x) = \frac{1}{\pi\lambda} \sqrt{\frac{thf}{\pi m\hbar\omega}}\, e^{\left(\frac{\Lambda^2}{2} - \frac{k_p^2}{2m\hbar\omega}\right) cthf} \cdot e^{\frac{\Lambda^2 k_p}{2mc\, th2f} sh\frac{i\lambda}{2}\partial_x} \cdot K_{2ix/\lambda}\left(\frac{\Lambda^2}{2} cth2f\right) \qquad (3.11)$$

In the nonrelativistic we obtain the expression for the standard-ordered d.f. of the nonrelativistic linear oscillator:

$$F^S(\chi, x) \to \frac{1}{\pi\hbar} \sqrt{\frac{1}{2} thf \cdot th2f} \cdot e^{\frac{ipx}{\hbar}\left(1 - \frac{1}{ch2f}\right)} \cdot e^{-\frac{1}{2}\left(\frac{p^2}{m\hbar\omega} + \frac{m\omega}{\hbar}x^2\right) th2f} \qquad (3.12)$$

[1] *M.Hillery, R.F. O'Conner, M.O.Scully, E.P. Wigner.* Phys.Rep.,1984, v.106, p.121.
[2] *H.W. Lee.* Phys.Rep. 1995, v.259, p.147.
[3] *J.E. Moyal.* Proc. Cambride Philos. Soc., 1949, v.45, p.99.
[4] *R.J. Glauder.* Phys.Rep.1963, v.13, p.62.
[5] *H.W. Lee, M.O. Scully.* Found.Phys.,1983,v.13,p.61.
[6] *L.Cohen.*J. Math. Phys.,1966, v7, p.781.
[7] *E.Wigner* . Phys.Rev., 1932, v.40, p.749.
[8] *C.L. Mehta.* J. Math. Phys., 1964, v.5, p.677.
[9] *R.W. Davies, K.T. R. Davies.* Ann. Phys., 1975, v.89, p.261.
[10] *E.A. Akhundova, V.V. Dodonov,V.I.Man'ko.* Physica, 1982, v.A115, p.215.
[11] *Sh.M. Nagiyev, E.I. Jafarov.* Fizika, 1997, v.3, N3.
[12] *N.M. Atakishiyev, R.M. Mir-Kasimov, Sh.M. Nagiyev.* Teor.1 Mat. B Fiz., 1980, v.44, p.47.

Ş.M.Nağıyev, E.İ. Cəfərov

QEYRİ-LOKAL RELYATİVİSTİK OSSİLYATOR ÜÇÜN FAZA-FƏZA PAYLANMA FUNKSİYALARI

Sonlu -fərq relyativistik kvant mexanikasının faza təsvirinə baxılmışdır. Qeyri-lokal relyativistik xətti ossilyator modeli üçün Viqner funksiyaları və standart nizamlanmış paylanma funksiyası hesablanmışdır. Viqner funksiyalarının köməkliyi ilə koordinatın, impulsun və enerjinin orta qiymətləri (Plank düsturu) tapılmışdır.

Ш.М. Нагиев, Э.И. Джафаров

ФАЗОВО-ПРОСТРАНСТВЕННЫЕ ФУНКЦИИ РАСПРЕДЕЛЕНИЯ НЕЛОКАЛЬНОГО РЕЛЯТИВИСТСКОГО ОСЦИЛЛЯТОРА

Рассмотрено фазовое представление конечно-разностной релятивистской квантовой механики. Вычислены для нелокальной релятивистской модели линейного осциллятора функции Вигнера и стандартно-упорядоченная функция распределения, а также их нерелятивистские пределы. Найдены средние значения координаты, импульса и энергии (формула Планка) с помощью функций Вигнера.

Дата поступления: 12.01.98 *Редактор: С.А. Алиев*