DIFFERENCE HARMONIC OSCILLATORS. I.

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A generalization of the usual quantum-mechanical factorization method for the case of difference Schrödinger equation is considered and a concept of a q-commutator is introduced.

1. Introduction

The purpose of this paper is the construction of difference models of the linear harmonic oscillator and we follow here the papers [1-8]. An oscillator is called harmonic when its oscillation frequency independents on its energy. In quantum mechanics this statement leads to its characterization by a differential Hamiltonian operator whose energy spectrum is discrete, lower-bound and equally spaced. If we relax the Strict Schrödinger quantization rule we find a family of harmonic oscillator models characterized by Hamiltonians that are difference operators.

In Section 2 we consider a simple difference generalization of the time-independent one-dimensional Schrödinger equation in the case of uniform lattice. As an example of the use of the difference Schrödinger equation we construct for one q- harmonic oscillator models (Section 5). The points of departure of our construction are a finite-difference generalization of the factorization method (Section 3 and 4) and the introduction of the concept of a q-commutator:

$$[A,B]_{q(x)}\equiv Aq(x)B-Bq^{-1}(x)A \tag{1.1}$$

The difference of Meixner oscillator model is introduced in Section 6, using well-known properties of the Meixner polynomials and its difference equation.

2. Difference Schrödinger equation

We consider the simplest case in which the time-independent one-dimensional Schrödinger equation

$$H_N\psi(x)=E\psi(x)\,,\quad H_N=-\frac{\hbar^2}{2m}\partial_x^2+V(x), \tag{2.1}$$

describing the motion of a particle in the potential field $V(x)$, is replaces by the finite-difference equation

$$-\frac{\hbar^2}{2m}\frac{\psi((x+h)-2\psi(x)+\psi((x-h)}{h^2}+V(x)\psi(x)=e\psi(x)\quad . \tag{2.2}$$

This equation approximates the Schrödinger equation (2.1) on a uniform lattice with step $\Delta x=h$ in the second order of accuracy. The Hamiltonian of eq. (2.2) is

$$H=-\frac{\hbar^2}{mh^2}(\cosh\partial_x-1)+V(x)\quad , \tag{2.3}$$

where it has definition $\exp(h\partial_x)f(x)=f(x+h)$.

The condition of hermiticity of the Hamiltonian H has the consequence that the step h of the finite-difference differentiation can be a purely imaginary of real quantity.

3. The quantum-mechanical factorization method

Let us consider the nonrelativistic one-dimensional Hamiltonian H_N (2.1) with the positive-definite and quadratic integrable wave function of the ground state $\psi_0=e^{-\rho(x)}$ and the energy $E_0: H_N\psi_0=E_0\psi_0$. We express $V(x)$ in terms of $\rho(x)$ and E_0:

$$V(x)=\frac{\hbar^2}{2m}[(\partial_x\rho)^2-\partial_x^2\rho]+E_0 \tag{3.1}$$

Then we can write down H_N in the factorized form $H_N-E_0=a^+a^-$, where

$$a^\pm=\frac{\hbar}{\sqrt{2m}}(\partial_x\rho\mp\partial_x)=\mp\frac{\hbar}{\sqrt{2m}}e^{\pm\rho(x)}\partial_x e^{\mp\rho(x)} \tag{3.2}$$

The operators $a^\pm$ are Hermitian conjugate with respect to the scalar product

$$(\psi,\phi)_1=\int_{-\infty}^{\infty}\psi^*(x)\phi(x)dx \tag{3.3}$$

and obey the commutation relation

$$[a-,a+]=\frac{\hbar^2}{m}\partial_x^2\rho \tag{3.4}$$

For the harmonic oscillator the commutator is constant:

$$[a^-,a^+]=\omega\hbar=const. \tag{3.5}$$

The solutions of this equation are

$$\rho(x) = \lambda x^2, V(x) = \frac{m\omega^2 x^2}{2}, \lambda = m\omega / 2\hbar \quad (3.6)$$

The creation and annihilation operators take the form

$$a^{\pm} = \frac{\hbar}{\sqrt{2m}}(2\lambda x \mp \partial_x).$$

The following relations

$$H_N = -\frac{\hbar^2}{2m}\partial_x^2 + \frac{m\omega^2 x^2}{2} = a^+a^- + \frac{\hbar\omega}{2}, \quad [H_N, a^{\pm}] = \pm a^{\pm}, a^-\psi_0 = 0 \quad (3.7)$$

are easily derived. They give us the method for the construction eigenvalues and the orthonormalized eigenfunctions of H_N:

$$En = \hbar\omega(n+1/2),$$

$$\psi_n(x) = c_n(a^+)^n e^{-\lambda x^2} c_n H_n = \sqrt{2\lambda x}\; e^{-\lambda x^2}, \quad (3.8)$$

$$c_n = \left(\frac{2\lambda}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}}, n = 0,1,2,\ldots,$$

where $H_n(x)$ are the Hermite polynomials and $(\psi_n, \psi_m)_1 = \delta_{nm}$.

4. The difference factorization method

Let us now generalize the factorization method to the case of the difference Schrödinger equation (2.2). We suppose that the ground-state wave function has again the form $\psi_0 = e^{-\rho(x)}$ and the energy e_0. Let us consider the difference operators

$$A^{\pm} = \mp h^{-1}\alpha(x)e^{\pm\rho(x)} \sinh\frac{h}{2}\partial_x e^{\mp\rho(x)}, \quad (4.1)$$

where $\alpha(x)$ and $\rho(x)$ are an arbitrary functions which we to define. These operators are Hermitian conjugate with respect to the scalar product

$$\psi, \phi)_2 \equiv \int_{-\infty}^{\infty} \psi^*(x)\phi(x)\frac{dx}{\alpha(x)} \quad (4.2)$$

In the case of difference operators (4.1) we have to consider, instead of the commutator (3.4), a more complicated q-commutator (1.1), where $q(x)$ is also an arbitrary function which we have to define. In the limit $h \to 0$ we have $\alpha(x) \to \alpha_0 = const$ and $q(x) \to 1$.

We calculate the q-commutator of the operators $A^{\pm}$:

$$A^-, A^+]_{q(x)} = \frac{1}{2h^2}\alpha(x)\left[\alpha\left(x + \frac{h}{2}\right)\sinh(\rho(x+h) - \rho(x)) - \alpha\left(x - \frac{h}{2}\right)\sinh(\rho(x) - \rho(x-h))\right] \quad (4.3)$$

where

$$q(x) = \exp\left[2\cosh\frac{h}{2}\partial_x\rho(x) - 2\rho(x)\right] \quad (4.4)$$

In the limit $h \to 0$

$$A^{\pm} \to \alpha_0(\partial_x\rho \mp \partial_x) = const \cdot a^{\pm}, \quad [A^-, A^+]_{q(x)} \to const \cdot [a^-, a^+]$$

and the expression in the right-hand side of (4.3) becames $\frac{1}{2}\alpha_0^2\partial_x^2\rho$.

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SONLU -FƏRQ HARMONİK OSSİLYATORLAR. I.

Adi kvant-mexaniki faktorizasiya üsulu sonlu-fərq Şredinqer tənliyi halına ümumiləşdirilmiş və q-kommutator anlayışı daxil edilmişdir.

DIFFERENCE HARMONIC OSCILLATORS. I.

Ш.М. Нагиев

РАЗНОСТНЫЕ ГАРМОНИЧЕСКИЕ ОСЦИЛЛЯТОРЫ. I.

Рассмотрено обобщение обычного квантово-механического метода факторизации на случай разностного уравнения Шредингера и введено понятие q-коммутатора.

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