SL(3,C) - SOLUTIONS OF SELF-DUALITY EQUATIONS

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New solutions of the Yang-Mill's self-duality equation are constructed by means of discrete symmetry transformations for the algebra SL(3,C). The obtained solutions are expressed in terms of determinants of special type matrixes.

1. Over the last few years self-dual Yang-Mills equation have attracted a fair amount or attention. It has been shown [1-8] that a large number of one, two and (1+2)-dimensional integrable models such as Korteweg-de Vries, N-waves, Ernst, Kadomtsev-Petviashvili, Toda lattice, nonlinear Schrodinger equations and many others can be obtained from the four-dimensional self-dual Yang-Mills equation by symmetry reduction and by imposing the constraints on Yang-Mills potentials.

The universality of the self-dual Yang-Mills (SDYM) model as an integrable system has been confirmed in the paper [9] where the general scheme of the reduction of the Belavin-Zakharov Lax pair for self-duality [10] has been represented over an arbitrary subgroup from the conformal group of transformations of R_4-space. As the result of this reduction one has the Lax pair representation for the corresponding differential equations of a lower dimension.

The problem of constructing of the instanton solutions in the explicit form for semisimple Lie algebra, rank of which is greater than two, remains also important for the present time.

Two effective methods of integration of SDYM for arbitrary semisimple algebra has been proposed in series of papers [11]. However, another approach has been suggested [12]. It based on discrete symmetry transformation that allows us to generate new solutions from the old ones. This method has been applied to many cases, for instance, the exact solutions of principal chiral field problem were obtained in [13].

The purpose of the present paper is to construct the exact solutions of the self-duality equations for the case of SL(3,C) algebra by means of discrete symmetry transformation method.

2. Self-dual equations are the systems of equations for the parameters of a group element G considering as the functions of four independent arguments $z, \bar{z}, y, \bar{y}$.

$$\left(G_{\bar{z}}G^{-1}\right)_z + \left(G_{\bar{y}}G^{-1}\right)_y = 0 \quad , \qquad (1)$$

where $G_t = \partial_t G$.

The system of equations (1) can be partially solved

$$G_{\bar{z}}G^{-1} = +f_y \ , \ G_{\bar{y}}G^{-1} = -f_z \quad ,$$

where the element f takes values in the algebra of corresponding group.

System of equations on f has the following form

$$f_{z\bar{z}} + f_{y\bar{y}} + [f_z, f_y] = 0 \qquad (2)$$

Following [12], for the case of a semisimple Lie algebra and for an element f being a solution of (2), the following statement takes place:

There exists such an element S taking values in a gauge group that

$$S^{-1}\frac{\partial S}{\partial y} = \frac{1}{\tilde{f}_-}\left[\frac{\partial \tilde{f}}{\partial y}, X_M\right] - \frac{\partial}{\partial \bar{z}}\frac{1}{\tilde{f}_-}X_M$$
$$S^{-1}\frac{\partial S}{\partial z} = \frac{1}{\tilde{f}_-}\left[\frac{\partial \tilde{f}}{\partial z}, X_M\right] + \frac{\partial}{\partial \bar{y}}\frac{1}{\tilde{f}_-}X_M \qquad (3)$$

Here X_N is the element of the algebra corresponding to its maximal root divided by its norm, i.e.,

$$\left[X_M^+, X^-\right] = H, \left[H, X^\pm\right] = \pm 2X^\pm \quad ,$$

$-\ \tilde{f}_-$ - is the coefficient function in the decomposition of $\tilde{f}$ of the element corresponding to the minimal root of the algebra, $\tilde{f} = \sigma f \sigma^{-1}$ and where σ is an automorfism of the algebra, changing the positive and negative roots.

In the case of algebra SL(3,C) we'll consider the case of three dimensional representation of algebra and the following form of $\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$.

The discrete symmetry transformation, producing new solutions from the known ones, is as follows:

$$\frac{\partial F}{\partial y} = S\frac{\partial \tilde{f}}{\partial y}S^{-1} + \frac{\partial S}{\partial \bar{z}}S^{-1} \quad , \quad \frac{\partial F}{\partial z} = S\frac{\partial \tilde{f}}{\partial z}S^{-1} - \frac{\partial S}{\partial \bar{y}}S^{-1} \qquad (4)$$

3. Let's represent the explicit formulae for transformation of the self-duality equations in the case of SL(2,C) algebra

$$f = \alpha_1 X_1^+ + \alpha_2 X_2^+ + \alpha_{1,2} X_{1,2}^+ + \tau_1 h_1 + \tau_2 h_2 + a_1 X_1^- + a_2 X_2^- + a_{1,2} X_{1,2}^- \quad , \qquad (5)$$

In connection with the general scheme, first of all, it is necessary to find the solution of the equations (3) for the SL(3,C) valued function S for given f, solution of self-duality equation (2).

From (3) it is clear that S is upper triangular matrix and can be represented in the following form:

$$S = \exp \beta_1 X_1^+ \exp \beta_{1,2} X_{1,2}^+ \exp \beta_2 X_2^+ \exp \beta_0 H \quad , \tag{6}$$

where $H=h_1+h_2$.

After substitution of the last representation of S into (3) and taking into account (5), we have at every step of the recurrent procedure the following relations

$$\begin{aligned} \beta_0 &= \ln \alpha_{1,2} , \quad \beta_1 = \alpha_2 , \quad \beta_2 = \alpha_1 \\ (\beta_{1,2})_y &= (\alpha_{1,2})_{\bar z} - (\delta_1 + \delta_2)_y \alpha_{1,2} - (\alpha_1)_y \alpha_2 \\ (\beta_{1,2})_z &= -(\alpha_{1,2})_{\bar y} - (\delta_1 + \delta_2)_z \alpha_{1,2} - (\alpha_1)_z \alpha_2 \end{aligned} \tag{7}$$

As the initial solution we'll take the explicit solution f belonging to the algebra of upper triangular matrixes:

$$f = \alpha_1 X_1^+ + \alpha_2 X_2^+ + \alpha_{1,2} X_{1,2}^+ + \tau_1 h_1 + \tau_2 h_2 \tag{8}$$

The component form of self-duality equations for this case is following

$$\begin{aligned} \Box \tau_i &= 0, \Box \alpha_i = \{\delta_i, \alpha_i\}_{y,z} \quad , \quad i=1,2, \\ \Box \alpha_{1,2} &= \{\delta_1 + \delta_2, \alpha_{1,2}\}_{y,z} \quad , \end{aligned} \tag{9}$$

where $\Box = \frac{\partial^2}{\partial z \partial \bar z} + \frac{\partial^2}{\partial y \partial \bar y}$; $\delta_1 = 2\tau_1 - \tau_2$, $\delta_2 = 2\tau_2 - \tau_1$

and figure brackets of two functions g_1 and g_2 denotes :

$$\{g_1, g_2\}_{y,z} = \frac{\partial g_1}{\partial y}\frac{\partial g_2}{\partial z} - \frac{\partial g_2}{\partial y}\frac{\partial g_1}{\partial z} .$$

The general solution of system (9) takes the form

$$\begin{aligned} \tau_i &= \oint_c \tau_i(\lambda) d\lambda \ , \quad \alpha_i = \oint_c \alpha_i(\lambda) \exp(-\bar\delta_i(\lambda)) d\lambda \ , \quad \bar\delta_i(\lambda) = \oint_c \frac{d\lambda' \delta_i(\lambda')) d\lambda'}{\lambda - \lambda'} , \quad i = 1,2 \ , \\ \alpha_{1,2} &= \oint_c \alpha_{1,2}(\lambda) \exp(-\bar\delta_1(\lambda) - \bar\delta_2(\lambda)) d\lambda + \oint_c \alpha_1(\lambda) \exp(-\delta_1(\lambda)) d\lambda \oint_c \frac{d\lambda' \alpha_2(\lambda') \exp(-\delta_2(\lambda'))}{\lambda - \lambda'} \end{aligned} \tag{10}$$

Here the circle integration goes over the complex parameter λ and all integrated functions are arbitrary functions of three independent variables $(y + \lambda \bar z, z - \lambda \bar y, \lambda)$.

By the direct check one can be convinced that (10) are the solutions of equations (9). The formulae (10) can also be obtained as a solution of homogeneous Riemann problem in the case of the solvable algebra [11].

Let's represent two types of Backlund transformation by means of which one can construct new types of solutions of equations (9) from the known solution (10). For solutions of first two equations of (9) this two Backlund transformations are the same:

$$\begin{aligned} (\alpha_i^k)_{\bar z} - (\delta_i)_y \alpha_i^k &= (\alpha_i^{k+1})_y \\ -(\alpha_i^k)_{\bar y} - (\delta_i)_z \alpha_i^k &= (\alpha_i^{k+1})_z \quad , i = 1,2 \end{aligned} \tag{11}$$

For solutions of the third equation of the system (9) they are different:

$$\begin{aligned} (\alpha_{1,2}^{0,k})_{\bar z} - (\delta_1 + \delta_2)_y \alpha_{1,2}^{0,k} - (\alpha_1^k)_y \alpha_2^k &= (\alpha_{1,2}^{0,k+1})_y \\ -(\alpha_{1,2}^{0,k})_{\bar y} - (\delta_1 + \delta_2)_z \alpha_{1,2}^{0,k} - (\alpha_1^k)_z \alpha_2^k &= (\alpha_{1,2}^{0,k+1})_z \end{aligned} \tag{12}$$

and

$$\begin{aligned} (\alpha_{1,2}^{k,0})_{\bar z} - (\delta_1 + \delta_2)_y \alpha_{1,2}^{k,0} + \alpha_1^k (\alpha_2^k)_y &= (\alpha_{1,2}^{k+1,0})_y \\ -(\alpha_{1,2}^{k,0})_{\bar y} - (\delta_1 + \delta_2)_z \alpha_{1,2}^{k,0} + \alpha_1^k (\alpha_2^k)_z &= (\alpha_{1,2}^{k+1,0})_z \end{aligned} \tag{13}$$

Note that starting, zero step of upper transformations procedure coincides with initial solutions (10).

Let's return to the solution of the equation (7) at the first step of the recurrent procedure.

Comparing (7) and (13) we came to the conclusion that $\beta_{1,2} = \alpha_{1,2}^{0,1}$.

Finally, knowing all components of matrix S and using (4) we can express the solution

$$F = F_1^+X_1^+ + F_2^+X_2^+ + F_{1,2}^+X_{1,2}^+ + F_1^0h_1 + F_2^0h_2 + F_1^-X_1^- + F_2^-X_2^- + F_{1,2}^-X_{1,2}^-$$

of self-duality equations at the first step of the recurrent procedure in terms of chains (11)-(13):

$$F_1^0 = \tau_1 + \frac{\alpha_{1,2}^{1,0}}{\alpha_{1,2}^{0,0}}, \quad F_2^0 = \tau_2 + \frac{\alpha_{1,2}^{0,1}}{\alpha_{1,2}^{0,0}}.$$

$$F_{1,2}^- = \frac{1}{\alpha_{1,2}^{0,0}}, \quad F_1^- = \frac{\alpha_2^0}{\alpha_{1,2}^{0,0}}, \quad F_2^- = -\frac{\alpha_1^0}{\alpha_{1,2}^{0,0}}$$

$$F_1^+ = -\frac{1}{\alpha_{1,2}^{0,0}}\begin{vmatrix}\alpha_1^0 & \alpha_1^1\\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{1,0}\end{vmatrix}, \quad F_2^+ = -\frac{1}{\alpha_{1,2}^{0,0}}\begin{vmatrix}\alpha_2^0 & \alpha_2^1\\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1}\end{vmatrix}$$

$$F_{1,2}^+ = \frac{1}{\alpha_{1,2}^{0,0}}\begin{vmatrix}\alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1}\\ \alpha_{1,2}^{1,0} & \alpha_{1,2}^{1,1}\end{vmatrix}$$

The general formulae of the recurrent procedure as well as the expression for the group element will be considered in further publication.

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[1] *R.S. Ward, Phil. Trans.* R. Soc. Lond.A315, 451 (1985); Lect. Notes Phys., 1987, 280, 106; Lond. Math. Soc. Lect. Notes Ser., 1990, 156, 246.

[2] *L.J. Mason and G.A. J.Sparling.* Phys. Lett., 1989, A137, 29; J. Geom. and Phys., 1992, 8, 243.

[3] *S. Chakravarty, M.J. Ablowitz and P.A. Clarkson.* Phys. Rev. Lett., 1990, 1085.

[4] *I. Bakas and D.A. Depireux.* Mod. Phys. Lett., 1991, A6, 399.

[5] *M.J. Ablowitz, S. Chakravarty and L.A. Takhtajan.* Comm. Math. Phys., 1993, 158, 1289.

[6] *T.A. Ivanova and A.D. Popov.* Phys. Lett., 1992, A170, 293.

[7] *L.J. Mason and N.M.J. Woodhouse.* Nonlinearity 1, 1988, 73; 1993, 6, 569.

[8] *M. Kovalyov, M. Legare and L. Gagnon.* J. Math. Phys., 1993, 34, 3425.

[9] *M. Legare and A.D. Popov.* Pis'ma Zh. Eksp. Teor. Fiz., 1994, 59, 845.

[10] *A.A. Belavin and V.E. Zakharov.* Phys. Lett., 1978, B73, 53.

[11] *A.N. Leznov and M.A. Mukhtarov.* J. Math. Phys., 1987, 28 (11), 2574; Prepr. IHEP, 1987, 87-90. Prepr. ICTP 163, Trieste, Italy, 1990; J. Sov. Lazer Research, 13 (4), 284, 1992.

[12] *A.N. Leznov.* IHEP preprint-92/87, 1990.

[13] *A.N. Leznov, M.A.Mukhtarov and W.J.Zakrzewski.* Tr. J. of Physics 1995, 19, 416.

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AVTODUAL TƏNLİYİNİN SL (3,C)-HƏLLƏRİ

Diskret simmetriya çevrilmələri metodu vasitəsilə SL(3,C) cəbri üçün Yanq-Mills avtodual tənliyinin yeni həlləri alınmışdır. Alınmış həllər xüsusi şəkilli matrisaların determinantların ifadə olunmuşdur.

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SL(3,C)-РЕШЕНИЯ УРАВНЕНИЙ АВТОДУАЛЬНОСТИ

Методом преобразований дискретных симметрий построены новые решения уравнений автодуальности Янга-Миллса для алгебры SL(3,C). Построенные решения выражены в терминах детерминантов матриц специального типа.

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