SL(3,C)-GROUP ELEMENT SOLUTION OF YANG-MILLS SELF-DUALITY

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The group element solutions of the Yang-Mill's self-duality equation are constructed by means of discrete symmetry transformations for the algebra SL(3,C).

1. The problem of constructing the instanton solutions of self-dual Yang-Mills equations in the explicit form remains important for the present time. This problem is solved only for the case of SL(2,C) algebra and for instanton number not greater than two. The famous ADHM ansatz [1] gives the qualitative description of instanton solution but not its explicit form. Two effective methods of integration of SDYM for arbitrary semisimple algebra has been proposed in series of papers [2]. Another, the discrete symmetry transformation approach has been suggested [3] that allows to generate new solutions from the old ones. This method has been applied to many cases, for instance, the exact solutions of principal chiral field problem were obtained in [4].

This work must be considered as a continuation of the paper [5] where the discrete symmetry transformation method has been applied for deriving the exact solution of Yang-Mills self-duality for the case of SL(3,C) algebra. The purpose of the present paper is to do the same for group-valued element what is important for applications.

2. Let us remind the basic statements from [5].

Self-dual equations are the systems of equations for the parameters of a group element G considering as the functions of four independent arguments $z, \bar{z}, y, \bar{y}$

$$\left(G_{\bar{z}}G^{-1}\right)_z + \left(G_{\bar{y}}G^{-1}\right)_y = 0 \ , \tag{1}$$

where $G_t=\partial_t G$.

The system of equations (1) can be partially solved

$$G_{\bar{z}}G^{-1} = f_y \ , \quad G_{\bar{y}}G^{-1} = -f_z \ , \tag{2}$$

where the element f takes values in the algebra of corresponding group.

System of equations on f has the following form

$$f_{z\bar{z}} + f_{y\bar{y}} + [f_z, f_y] = 0 \tag{3}$$

Following [3], for the case of a semisimple Lie algebra and for an element f being a solution of (2), the following statement takes place:

There exists such an element S taking values in a gauge group that

$$S^{-1}\frac{\partial S}{\partial y} = \frac{1}{\tilde{f}_-}\left[\frac{\partial \tilde{f}}{\partial y}, X_M\right] - \frac{\partial}{\partial \bar{z}}\frac{1}{\tilde{f}_-}X_M$$

$$S^{-1}\frac{\partial S}{\partial z} = \frac{1}{\tilde{f}_-}\left[\frac{\partial \tilde{f}}{\partial z}, X_M\right] + \frac{\partial}{\partial \bar{y}}\frac{1}{\tilde{f}_-}X_M \tag{4}$$

Here X_M is the element of the algebra corresponding to its maximal root divided by its norm, $-\ \tilde{f}_-$ - is the coefficient function in the decomposition of $\tilde{f}$ of the element corresponding to the minimal root of the algebra, $\tilde{f} = \sigma f \sigma^{-1}$ and where σ is an automorfism of the algebra, changing the positive and negative roots.

In the case of algebra SL(3,C) we'll consider the case of three dimensional representation of algebra and the following form of $\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$.

The discrete symmetry transformation, producing new solutions from the known ones, is as follows:

$$\frac{\partial F}{\partial y} = S\frac{\partial \tilde{f}}{\partial y}S^{-1} + \frac{\partial S}{\partial \bar{z}}S^{-1} \ , \frac{\partial F}{\partial z} = S\frac{\partial \tilde{f}}{\partial z}S^{-1} - \frac{\partial S}{\partial \bar{y}}S^{-1} \tag{5}$$

Using (2) the relations (5) can be rewritten in terms of group-valued element as

$$\left(S_n\sigma g_n\right)_{\bar{z}}\left(S_n\sigma g_n\right)^{-1} = \left(f_{n+1}\right)_y \ , \ \left(S_n\sigma g_n\right)_{\bar{y}}\left(S_n\sigma g_n\right)^{-1} = -\left(f_{n+1}\right)_z \ ,$$

where

$$\left(g_n\right)_{\bar{z}}g_n^{\ -1} = \left(f_n\right)_y \ , \ \left(g_n\right)_{\bar{y}}g_n^{\ -1} = -\left(f_n\right)_z$$

So we see that the group valued elements g_{n+1} and g_n are connected by the relation

$$g_{n+1} = S_n\sigma g_n \tag{6}$$

3. Let's represent the explicit formulae of the recurrent procedure of obtaining the group-valued element solutions of the self-duality equations in the case of SL(2,C) algebra.

At every step, as it shown in [5], S is upper triangular matrix and can be represented in the following form:

$$S_n = \exp(\beta_1)_n X_1^+ \exp(\beta_{1,2})_n X_{1,2}^+ \exp(\beta_2)_n X_2^+ \exp(\beta_0)_n H \quad , \tag{7}$$

where $H=h_1+h_2$ and for g_n we use the following parameterization:

$$g_n = \exp(\eta_1^+)_n X_1^+ \exp(\eta_{1,2}^+)_n X_{1,2}^+ \exp(\eta_2^+)_n X_2^+ \exp((t_1)_n h_1 + (t_2)_n h_2) \times \\ \times \exp(\eta_2^-)_n X_2^- \exp(\eta_{1,2}^-)_n X_{1,2}^- \exp(\eta_1^-)_n X_1^- \tag{8}$$

with

$$g_0 = \exp(\eta_1^+)_0 X_1^+ \exp(\eta_{1,2}^+)_0 X_{1,2}^+ \exp(\eta_2^+)_0 X_2^+ \exp((t_1)_0 h_1 + (t_2)_0 h_2)$$

as an initial solution.

Hereafter, $X_1^\pm, X_2^\pm, X_{1,2}^\pm, h_1, h_2$ are the generators of SL(3,C) algebra.

Following the general scheme from [5] and using (2) and (6) we have at

(0)-step:

$$(t_i)_0 = \tau_i^{-1} \equiv \nu_i \,,\, (\eta_i^+)_0 = \alpha_i^{-1} \,,\, i = 1,2 \,,\, (\eta_{1,2}^+)_0 = \alpha_{1,2}^{-1,0} \quad ;$$

(1)-step:

$$(t_1)_1 = -\nu_1 + \ln\left(-\frac{\alpha_{1,2}^{-1,0}}{\alpha_{1,2}^{0,0}}\right) \,,\quad (t_2)_1 = -\nu_2 + \ln\left(-\frac{\alpha_{1,2}^{0,-1}}{\alpha_{1,2}^{0,0}}\right) ,$$

$$(\eta_1^-)_1 = -\frac{\alpha_2^{-1}}{\alpha_{1,2}^{-1,0}} \exp\delta_1 \,,\quad (\eta_2^-)_1 = \frac{\alpha_1^{-1}}{\alpha_{1,2}^{0,-1}} \exp\delta_2 \,,\quad (\eta_{1,2}^-)_1 = \frac{1}{\alpha_{1,2}^{-1,0}} \exp(\delta_1 + \delta_2),$$

$$(\eta_1^+)_1 = -\frac{\det\begin{pmatrix}\alpha_1^{-1} & \alpha_1^0 \\ \alpha_{1,2}^{-1,0} & \alpha_{1,2}^{0,0}\end{pmatrix}}{\alpha_{1,2}^{-1,0}} \,,\quad (\eta_2^+)_1 = -\frac{\det\begin{pmatrix}\alpha_2^{-1} & \alpha_2^0 \\ \alpha_{1,2}^{0,-1} & \alpha_{1,2}^{0,0}\end{pmatrix}}{\alpha_{1,2}^{0,-1}} \,,\quad (\eta_{1,2}^+)_1 = \frac{\det\begin{pmatrix}\alpha_{1,2}^{-1,0} & \alpha_{1,2}^{-1,1} \\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1}\end{pmatrix}}{\alpha_{1,2}^{-1,0}} ,$$

$$\delta_i = 2\nu_i - \nu_j \,,\, i \neq j \quad ;$$

(2)-step:

$$(\eta_1^-)_2 = -\frac{\det\begin{pmatrix}\alpha_2^{-1} & \alpha_2^0 \\ \alpha_{1,2}^{1,-1} & \alpha_{1,2}^{1,0}\end{pmatrix}}{\det\begin{pmatrix}\alpha_{1,2}^{-1,0} & \alpha_{1,2}^{-1,1} \\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1}\end{pmatrix}} \exp\delta_1 \,,\quad (\eta_2^-)_2 = \frac{\det\begin{pmatrix}\alpha_1^{-1} & \alpha_1^0 \\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{1,0}\end{pmatrix}}{\det\begin{pmatrix}\alpha_{1,2}^{0,-1} & \alpha_{1,2}^{0,0} \\ \alpha_{1,2}^{1,-1} & \alpha_{1,2}^{1,0}\end{pmatrix}} \exp\delta_2 \quad ,$$

$$(\eta_{1,2}^-)_2 = \frac{1}{\det\begin{pmatrix}\alpha_{1,2}^{-1,0} & \alpha_{1,2}^{-1,1} \\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1}\end{pmatrix}} \exp(\delta_1 + \delta_2),$$

$$(\eta_1^+)_2 = \frac{\det\begin{pmatrix}\alpha_1^{-1} & \alpha_1^0 & \alpha_1^1 \\ \alpha_{1,2}^{-1,0} & \alpha_{1,2}^{0,0} & \alpha_{1,2}^{1,0} \\ \alpha_{1,2}^{-1,1} & \alpha_{1,2}^{0,1} & \alpha_{1,2}^{1,1}\end{pmatrix}}{\det\begin{pmatrix}\alpha_{1,2}^{-1,0} & \alpha_{1,2}^{-1,1} \\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1}\end{pmatrix}_1} \,,\quad (\eta_2^+)_2 = \frac{\det\begin{pmatrix}\alpha_2^{-1} & \alpha_2^0 & \alpha_2^1 \\ \alpha_{1,2}^{0,-1} & \alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1} \\ \alpha_{1,2}^{1,-1} & \alpha_{1,2}^{1,0} & \alpha_{1,2}^{1,1}\end{pmatrix}}{\det\begin{pmatrix}\alpha_{1,2}^{0,-1} & \alpha_{1,2}^{0,0} \\ \alpha_{1,2}^{1,-1} & \alpha_{1,2}^{1,0}\end{pmatrix}_2} ,$$

$$\left(\eta^{+}_{1,2}\right)_2 = \frac{\det\begin{pmatrix} \alpha_{1,2}^{-1,0} & \alpha_{1,2}^{-1,1} & \alpha_{1,2}^{-1,2} \\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1} & \alpha_{1,2}^{0,2} \\ \alpha_{1,2}^{1,0} & \alpha_{1,2}^{1,1} & \alpha_{1,2}^{1,2} \end{pmatrix}}{\det\begin{pmatrix} \alpha_{1,2}^{-1,0} & \alpha_{1,2}^{-1,1} \\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1} \end{pmatrix}}$$

Here, α_1^i, α_2^j, $\alpha_{1,2}^{i,j}$ - chains of solutions of self-duality equations determined by formulae (10-13) from [5].

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SL(3,C)-QRUP ELEMENTİ ÜÇÜN YANQ-MİLLS AVTODUALLIĞIN HƏLLİ

SL(3,C)-cəbri hallında diskret spektr metodu vasitəsilə qrup elementi üçün avtoduallığın Yanq-Mills tənliyinin həlləri tapılmışdır.

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РЕШЕНИЕ АВТОДУАЛЬНОСТИ ЯНГА-МИЛЛСА ДЛЯ SL(3,C)-ГРУППОВОГО ЭЛЕМЕНТА

Построены решения для группового элемента уравнений автодуальности Янга-Миллса методом дискретных симметрий в случае алгебры SL(3,C).

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