DIFFERENCE HARMONIC OSCILLATORS. II.

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Four models of a q-harmonic oscillator with wave functions expressed in terms of q-orthogonal polynomials are constructed.

5. Models of q-harmonic oscillator

In the previous paper [1] we are developed the factorization method to the case of the difference Schrodinger equation. The purpose of this paper is to construct the models of q-harmonic oscillator. Models of q-harmonic oscillator are being developed in connection with quantum groups (see, for example [2-11]). By a q-harmonic oscillator we understand a physical system described by an associative algebra with an identity generated by three generating operators of q-creation b^+, q-annihilation $b^-$and "particle number" N. They satisfy to the q- Heisenberg algebra

$$[b^-, b^+]_q = q, \quad [N, b^{\pm}]_q = \pm b^{\pm} \tag{5.1}$$

or

$$b^-b^+ - q_0 b^+b^- = 1,$$

where $q_0=q^{-2}$. The Hamiltonian for the q-oscillator can be defined as $H=b^+b^-$, then from the expression (5.1) we have:

$$[H, b^+]_q = qb^+, \quad [b^-, H]_q = qb^-, \quad [b^-, b^+] = 1 - (1 - q_0)H \equiv q_0^N. \tag{5.2}$$

The equation that describes the q-harmonic oscillator has the following form

$$H\Psi_n \equiv \frac{1 - q_0^N}{1 - q_0}\Psi_n = e_n\Psi_n, \qquad b^-\Psi_n = e_n^{\frac{1}{2}}\Psi_{n-1}, \qquad b^+\Psi_n = e_{n+1}^{\frac{1}{2}}\Psi_{n+1}, \tag{5.3}$$

where the energy spectrum is equal to

$$e_n = \frac{1 - q_0^n}{1 - q_0}, \qquad n = 0,1,2,\ldots \tag{5.4}$$

In order to realize explicitly the operators $b^{\pm}$ it is natural by to consider the situation

$$[A^-, A^+]_{q(x)} = const \tag{5.5}$$

by analogy with (3.5) [1] as the q-harmonic oscillator.

5.1. First we consider the case when h is the real quantity. Considering (5.5) as the equation for $\rho(x)$ we find that $\rho(x)=\lambda x^2$, which means that the ground state wave function for the q-harmonic oscillator model coincides with the non-relativistic one. In this case we have

$$q(x) = e^{\frac{\lambda h^2}{2}} = e^{\frac{\kappa}{2}}, \; \alpha(x) = \lambda^{-\frac{1}{2}}(ch\lambda hx)^{-1} \tag{5.6}$$

i.e. $q(x)$ is a constant. Using $\rho(x)=\lambda x^2$ and (5.6) we find:

$$[A^-, A^+]_q = qA^-A^+ - q^{-1}A^+A^- = \kappa^{-1}(q - q^{-1}) \tag{5.7}$$

It is easy to see that the q-creation and q-annihilation operators $b^{\pm}$ are connected with $A^{\pm}$ as follows:

$$b^{\pm} = q\sqrt{\frac{æ}{q^2 - 1}}A^{\pm} = \mp\frac{q_0^{\pm 1/4}}{2\sqrt{1 - q_0}chæt}\left(e^{\mp æ t}e^{\frac{1}{2}\partial_t} - e^{\pm æ t}e^{-\frac{1}{2}\partial_t}\right), \tag{5.8}$$

where $x=ht$.

It can be shown that the orthonormalized wave functions of this q-oscillator model are expressed by means of q^{-1} - Hermite polynomials $h_n(x/q)$:

$$\psi_n(x) = c_n h_n\left(sh\lambda hx|q_0\right) e^{-\lambda x^2} \tag{5.9}$$

The condition $(\psi_n, \psi_m)_2=\delta_{nm}$ gives us the orthogonality condition on the complete real axis for q^{-1} - Hermite polynomials

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}h_n\left(sh\alpha x|q\right)\, h_m\left(sh\alpha x|q\right)\, ch\alpha x dx = q^{\frac{-n(n+1)}{2}-\frac{1}{8}}\left(q;q\right)_n \delta_{nm}, \quad q = e^{-2\alpha^2}. \tag{5.10}$$

Using the following limit relation

$$\lim_{\alpha \to 0} \alpha^{-n} \; h_n(sh\alpha x|q) = H_n(x),$$

which is readily proved by induction, we see that the wave functions (5.10) in the limit $h \to 0$ (i.e., as $q_0 \to 1^-$) transform to the wave functions of the nonrelativistic linear harmonic oscillator (3.8) [1].

5.2. We now consider the case when h is imaginary quantity: $h = i\delta$. In this case the solutions of the equation (5.5) have the following form:

$$\rho(x) = \lambda x^2 \;, \; q(x) = q = e^{-\frac{\lambda\delta^2}{2}} = e^{-\frac{æ}{2}} \;, \; \alpha(x) = \lambda^{-1/2} \left(\cos \lambda\delta x\right)^{-1} \tag{5.11}$$

The q- commutator of the operators $A^{\pm}$ is equal to

$$\left[A^-, \; A^+\right]_q = æ^{-1}\left(q^{-1} - q\right) \tag{5.12}$$

For the q-creation and q-annihilation operators we have the following expressions:

$$b^{\pm} = q\sqrt{\frac{æ}{1 - q^2}} \quad A^{\pm} = \pm\, i \,\frac{q_\circ^{\pm 1/4}}{2\sqrt{q_\circ - 1}\;\cos æt}\left(e^{\mp æ t} e^{\frac{i}{2}\partial_t} - e^{\pm æ t} e^{-\frac{i}{2}\partial_t}\right), \tag{5.13}$$

where $x = \delta t$.

Wave functions of this q-oscillator model are expressed in terms of the q-Hermite polynomials $H_n(x|q)$:

$$\psi_n(x) = c_n H_n\left(\sin \lambda\delta x \middle| q_0^{-1}\right) e^{-\lambda x^2} \tag{5.14}$$

The q-Hermite polynomials are defined by the recurrence relation

$$H_{n+1}(x|q) = 2x\, H_n(x|q) - \left(1 - q^n\right) H_{n-1}(x|q) \quad , \qquad 0 < q < 1.$$

The orthogonality condition follows from $(\psi_n, \psi_m)_2 = \delta_{nm}$ for the q-Hermite polynomials:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_n(\sin \alpha x|q)\; H_m(\sin \alpha x|q)\; \cos \alpha x dx = q^{\frac{1}{8}}(q; q)_n \,\delta_{nm} \tag{5.15}$$

5.3. For the construction of the third model of the q-harmonic oscillator, one can consider the following realization of the operators $b^{\pm}$:

$$b^{\pm} = \pm \frac{i}{\sqrt{1 - q_\circ}}\, e^{\mp \lambda x^2}\left(e^{\mp 2i\lambda\delta x} - \sqrt{q_\circ}\; e^{-\frac{i\delta}{2}\partial_x}\right) e^{\pm \lambda x^2} \tag{5.16}$$

where $q_\circ = e^{-\lambda\delta^2}$. The operators (5.16) are Hermite conjugate in respect of the scalar product $(\psi, \phi)_1$ (3.3) [1].

The wave functions in this case are expressed by means of the Rogers-Szego polynomials $H_n(x/q)$:

$$\psi_n(x) = c_n H_n\left(- q^{\frac{-1}{2}} \; e^{-2i\lambda\delta x}; \; q_\circ\right) e^{-\lambda x^2} \tag{5.17}$$

The Rogers-Szego polynomials are defined as follows:

$$H_n(x; q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficient [12]

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, \qquad (a; q)_n = \prod_{j=0}^{n-1}(1 - aq^j)$$

It follows from the condition $(\psi_n, \psi_m)_2 = \delta_{nm}$ that the Rogers-Szego polynomials satisfy to the orthogonality relation on the full real line

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_n\left(- q^{-1/2} e^{2i\alpha x}; q\right) H_m\left(- q^{-1/2} e^{-2i\alpha x}; q\right) dx = q^{-n}(q; q)_n \delta_{nm} \;, \tag{5.18}$$

where $q = e^{-2\alpha^2}$.

We can also prove the following limit relation:

$$\lim_{\alpha \to 0}\left(-i\sqrt{\frac{2q}{1-q}}\right)^n H_n\left(-q^{-1/2}e^{-2i\alpha x};q\right) = H_n(x). \tag{5.19}$$

5.4. The fourth model of q-oscillator can be constructed if we take the following realization of q-creation and q-annihilation operators

$$b^{\pm} = \pm\frac{i}{\sqrt{1-q_0}}\left(e^{\pm\delta\partial_x} - q_0^{\frac{1}{2}\pm\frac{1}{4}}e^{\lambda\delta x}e^{\pm\frac{1}{2}\delta\partial_x}\right) \tag{5.20}$$

The operators (5.20) are Hermite conjugate in respect of the scalar product $(\psi,\phi)_1$ (3.3) [1].

The wave functions in this model have the form

$$\psi_n(x) = c_n h_n\left(-q_0^{-\frac{1}{2}}\ e^{-2\lambda\delta x};\ q_0\right) e^{-\lambda x^2},\quad q_0 = e^{-\lambda\delta^2}, \tag{5.21}$$

where $h_n(x/q)$ are the Stieltjes-Vigert polynomials:

$$h_n(x;q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2}x^k,$$

We can prove also that

$$\lim_{\alpha \to 0}\left(\frac{2q}{1-q}\right)^{n/2} h_n\left(-q^{-1/2}e^{-2\alpha x};q\right) = H_n(x), \tag{5.22}$$

where $q = e^{-2\alpha^2}$.

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SONLU-FƏRQ HARMONİK OSSİLYATORLAR. II.

Dalğa funksiyaları q-ortoqonal çoxhədlilərlə ifadə olunan q-harmonik ossilyatorun dörd modeli qurulmuşdur.

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РАЗНОСТНЫЕ ГАРМОНИЧЕСКИЕ ОСЦИЛЛЯТОРЫ. II.

Построены четыре модели q-гармонического осциллятора, волновые функции которых выражаются через q—ортогональные полиномы.

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