SOLUTIONS OF THE PRINCIPAL CHIRAL FIELD PROBLEM FOR THE HIGH-RANK SIMPLE ALGEBRAS

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New solutions of the principal chiral field problem are constructed by means of discrete symmetry transformations for the algebra SL(3,C). The generalization to the case of arbitrary semisimple algebra of the rank higher than two is discussed.

1. The problem of constructing of the solutions of self-dual Yang-Mills (SDYM) model and its dimensional reductions, the principal chiral field problem in our case, in the explicit form for semisimple Lie algebra, rank of which is greater than two, remains important for the present time. The interest arises from the fact that almost all integrable models in one, two and (1+2)-dimensions are symmetry reductions of SDYM or they can be obtained from it by imposing the constraints on Yang-Mills potentials [1-10].

This work is a direct continuation of [13], where the exact solutions of the principal chiral field problem have been derived for the case of algebra SL(2,C). The discrete symmetry transformation method [12] applied here allows to generate new solutions from the old ones in much more easier way than applying methods from [11], and the case of SL(3,C) algebra gives us a key to construct solutions for an arbitrary semisimple algebra.

2. Equations of the principal chiral field problem are the systems of equations for the element f, taking values in the semisimple algebra,

$$(q_i - q_j)\frac{\partial^2 f}{\partial x_i \partial x_j} = \left[\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}\right] \qquad (1)$$

In the case of two-dimensional space: $q_1=1$, $q_2=-1$, $x_1 = x$, $x_2 = n$.

Following [12], for the case of a semisimple Lie algebra and for an element f being a solution of (1), the following statement takes place:

There exists such an element S taking values in a gauge group that

$$S^{-1}\frac{\partial S}{\partial x_i} = \frac{1}{\tilde{f}_-}\left[\frac{\partial \tilde{f}}{\partial x_i}, X_M^+\right] - q_i \frac{\partial}{\partial x_i}\frac{1}{\tilde{f}_-} X_M^+ \qquad (2)$$

Here X_M^+ is the element of the algebra corresponding to its maximal root divided by its norm, i.e.,

$$\left[X_M^+, X_M^-\right] = H, \left[H, X_M^\pm\right] = \pm 2 X_M^\pm \quad ,$$

$-\tilde{f}_-$ - is the coefficient function in the decomposition of $\tilde{f}$ of the element corresponding to the minimal root of the algebra, $\tilde{f} = sfs^{-1}$ and where s is an automorfism of the algebra, changing the positive and negative roots.

In the case of algebra SL(3,C) we'll consider the case of three dimensional representation of algebra and the following form of $s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$.

The discrete symmetry transformation, producing new solutions from the known ones, is as follows:

$$\frac{\partial F}{\partial x_i} = S\frac{\partial \tilde{f}}{\partial x_i}S^{-1} + q_i \frac{\partial S}{\partial x_i}S^{-1} \qquad (3)$$

3. Let's represent the explicit formulae for transformation in the case of SL(3,C) algebra

$$f = a_1 X_1^+ + a_2 X_2^+ + a_{1,2} X_{1,2}^+ + t_1 h_1 + t_2 h_2 + a_1 X_1^- + a_2 X_2^- + a_{1,2} X_{1,2}^- \quad , \qquad (4)$$

In connection with the general scheme, first of all, it is necessary to find the solution of the equations (2) for the SL(3,C) valued function S for given f, solution of equations (1).

From (2) it is clear that S is upper triangular matrix and can be represented in the following form:

$$S = \exp b_1 X_1^+ \exp b_{1,2} X_{1,2}^+ \exp b_2 X_2^+ \exp b_0 H \quad , \qquad (5)$$

where $H=h_1+h_2$.

After substitution of the last representation of S into (2) and taking into account (4), we have at every step of the recurrent procedure the following relations

$$b_0 = \ln a_{1,2}\,,\ b_1 = a_2\,,\ b_2 = a_1$$

$$(b_{1,2})_{x_i} = (a_{1,2})_{x_i} - (d_1 + d_2)_{x_i} a_{1,2} - (a_1)_{x_i} a_2 \qquad (6)$$

As the initial solution we'll take the explicit solution f belonging to the algebra of upper triangular matrixes:

$$f = a_1 X_1^+ + a_2 X_2^+ + a_{1,2} X_{1,2}^+ + t_1 h_1 + t_2 h_2 \qquad (7)$$

The component form of self-duality equations for this case is following

$$\frac{\partial^2 t_n}{\partial x_i \partial x_j} = 0\,,$$

$$\frac{\partial^2 a_n}{\partial x_i \partial x_j} = \{d_n, a_n\}_{x_i, x_j}\,, \quad i=1,2, \qquad (8)$$

$$\frac{\partial^2 a_{1,2}}{\partial x_i \partial x_j} = \{ d_1 + d_2 , a_{1,2} \}_{x_i, x_j} ,$$

where $d_1 = 2t_1 - t_2 , d_2 = 2t_2 - t_1$ and figure brackets of two functions g_1 and g_2 denotes:

$$\{ g_1 , g_2 \}_{x_i, x_j} = \frac{\partial g_1}{\partial x_i}\frac{\partial g_2}{\partial x_j} - \frac{\partial g_2}{\partial x_j}\frac{\partial g_1}{\partial x_i} .$$

The general solution of system (8) takes the form

$$t_i = \sum_{s=1} t_i^s (x_s) , \; a_i = \oint_c a_i (l) \exp(-\bar{d}_i (l)) d l ,$$

$$\bar{d}_i (l) = \sum_{s=1} \frac{t_i^s (x_s)}{l + q_s} ,$$

$$a_{1,2} = \oint_c a_{1,2}(l) \exp(-\bar{d}_1(l) - \bar{d}_2(l)) dl + \oint_c a_1(l) \exp(-\bar{d}_1(l)) dl \oint_c \frac{dl' a_2(l') \exp(-\bar{d}_2(l'))}{l - l'} \tag{9}$$

Here the circle integration goes over the complex parameter l.

By the direct check one can be convinced that (9) are the solutions of equations (8). The formulae (9) can also be obtained as a solution of homogeneous Riemann problem in the case of the solvable algebra [11].

Let's represent two types of Backlund transformation by means of which one can construct new types of solutions of equations (8) from the known solution (9). For solutions of first two equations of (8) this two Backlund transformations are the same:

$$q_s (a_i^k)_{x_s} - (d_i)_{x_s} a_i^k = (a_i^{k+1})_{x_s} \; , i = 1,2 \tag{10}$$

For solutions of the third equation of the system (8) they are different:

$$q_s (a_{1,2}^{0,k})_{x_s} - (d_1 + d_2)_{x_s} a_{1,2}^{0,k} - (a_1^k)_{x_s} a_2^k = (a_{1,2}^{0,k+1})_{x_s} \tag{11}$$

and

$$q_s (a_{1,2}^{k,0})_{x_s} - (d_1 + d_2)_{x_s} a_{1,2}^{k,0} - a_1^k (a_2^k)_{x_s} = (a_{1,2}^{k+1,0})_{x_s} \tag{12}$$

Note that starting, zero step of upper transformations procedure coincides with initial solutions (9).

Let's return to the solution of the equation (7) at the first step of the recurrent procedure.

Comparing (6) and (12) we came to the conclusion that $b_{1,2} = a_{1,2}^{0,1}$.

Finally, knowing all components of matrix S and using (3) we can express the solution

$$F = F_1^+ X_1^+ + F_2^+ X_2^+ + F_{1,2}^+ X_{1,2}^+ + F_1^0 h_1 + F_2^0 h_2 + F_1^- X_1^- + F_2^- X_2^- + F_{1,2}^- X_{1,2}^-$$

of self-duality equations at the first step of the recurrent procedure in terms of chains (10)-(12):

$$F_1^0 = t_1 + \frac{a_{1,2}^{1,0}}{a_{1,2}^{0,0}} , F_2^0 = t_2 + \frac{a_{1,2}^{0,1}}{a_{1,2}^{0,0}}$$

$$F_{1,2}^- = \frac{1}{a_{1,2}^{0,0}} , F_1^- = \frac{a_2^0}{a_{1,2}^{0,0}} , F_2^- = -\frac{a_1^0}{a_{1,2}^{0,0}}$$

$$F_1^+ = -\frac{1}{a_{1,2}^{0,0}} \begin{vmatrix} a_1^0 & a_1^1 \\ a_{1,2}^{0,0} & a_{1,2}^{1,0} \end{vmatrix} , \; F_2^+ = -\frac{1}{a_{1,2}^{0,0}} \begin{vmatrix} a_2^0 & a_2^1 \\ a_{1,2}^{0,0} & a_{1,2}^{0,1} \end{vmatrix}$$

$$F_{1,2}^+ = \frac{1}{a_{1,2}^{0,0}} \begin{vmatrix} a_{1,2}^{0,0} & a_{1,2}^{0,1} \\ a_{1,2}^{1,0} & a_{1,2}^{1,1} \end{vmatrix}$$

The general formulae of the recurrent procedure as well as the expression for the group element will be considered in further publication.

As it is seen from formulas (11-12) for algebras of the rank higher than two, the number of corresponding Backlund transformations of the initial problem solutions will be equal to the rank of the algebra. Thus, it is necessary only to overcome the routine calculations using, for example, Mathematica 4-0 software.

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