More on Noncommutative Superstring Worldsheet

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Abstract

In this paper some properties of the superstring with noncommutative worldsheet are studied. We study the noncommutativity of the spacetime, generalization of the Poincaré symmetry of the superstring, the changes of the metric, antisymmetric tensor and dilaton.

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1 Introduction

The noncommutative geometry [1] has been considered for some time in connection with various physics subjects. Recent motivation to study the noncommutative geometry mainly comes from the string theory. String theories have been pointing towards a noncommuting scenario already in the 80's [2]. Recently Yang-Mills theories on noncommutative spaces have emerged in the context of M-theory compactified on a torus in the presence of constant background three-form field [3], or as low-energy limit of open strings in a background B-field, describing the fluctuations of the D-brane worldvolume [4, 5, 6]. The worldvolume of a D-brane with constant background B-field, is an example of a noncommutative spacetime, in which gauge and matter fields live [3, 4, 5, 7]. In other side, fundamental strings in the R⊗R background B-field become noncommutative. This was shown in the context of matrix theory [8].

Field theory on a noncommutative space has been proved to be useful in understanding various physical phenomena. Noncommutative field theory means that fields are thought of as functions over noncommutative space. At the algebraic level, the fields become operators acting on a Hilbert space as a representation space of the noncommutative space [9].

Previously we considered the superstring action as a two dimensional noncommutative field theory [10]. Up to the first order of the noncommutativity parameter, and some additional terms to the noncommutative superstring action, some physical quantities are obtained. For example, we obtained new supersymmetric action for string, extended boundary state of closed superstring, new boundary conditions for open string which lead to the generalization of the noncommutativity parameter of the spacetime.

In this paper, we do not consider the additional terms to the superstring action. According to this, now we study some other properties of the superstring with noncommutative worldsheet, to all orders of the noncommutativity parameter. Noncommutative worldsheet of string gives a noncommutative spacetime. If some directions of the spacetime are compacted on tori, the vacuum expectation value of the spacetime noncommutativity parameter can be non-zero. Noncommutative worldsheet also enables us to generalize the Poincaré symmetry. That is, some additional terms can be added to the ordinary Poincaré transformations. Therefore, we obtain a generalized conserved current. Furthermore, the NS⊗NS fields of superstring modify by a common phase. Making use of the Refs.[10, 11], one can find descriptions of the noncommutativity parameter of the worldsheet. For $\theta = 0$, all these results reduce to the known cases of the superstring theory with ordinary worldsheet, as expected.

This paper is organized as follows. In section 2, we study the action of the superstring with noncommutative worldsheet. In section 3, the noncommutativity of the spacetime, extracted from the noncommutativity of the string worldsheet, is studied. In section 4, a general form of the Poincaré symmetry is given. In section 5, the effects of the noncommutativity of the worldsheet on the metric, Kalb-Ramond field and dilaton are extracted.

2 Noncommutative worldsheet

We look at the superstring action as a two dimensional noncommutative field theory. In other words, assume that the superstring worldsheet is a two dimensional noncommutative space. In this space superstring with worldsheet supersymmetry has the action

$$S_* = -\frac{1}{4\pi\alpha'}\int d^2\sigma\left(\partial_a X^\mu * \partial^a X_\mu - i\bar{\psi}^\mu * \rho^a \partial_a \psi_\mu\right) , \tag{1}$$

where the spacetime and the worldsheet metrics are $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, ..., 1)$ and $\eta_{ab} = \mathrm{diag}(-1, 1)$. We later discuss about the supersymmetry of this action. The star product in this action is defined between any two functions of the worldsheet coordinates $\sigma^a = (\sigma, \tau)$

$$f(\sigma, \tau) * g(\sigma, \tau) = \exp\left(\frac{i}{2}\theta^{ab}\frac{\partial}{\partial\zeta^a}\frac{\partial}{\partial\eta^b}\right)f(\zeta^0, \zeta^1)g(\eta^0, \eta^1)\Big|_{\zeta^a=\eta^a=\sigma^a} . \tag{2}$$

The antisymmetric tensor θ^{ab} has only one independent component i.e., $\theta^{ab} = \theta\epsilon^{ab}$, where $\epsilon^{01} = -\epsilon^{10} = 1$. Definition (2) gives the noncommutativity between the worldsheet coordinates as

$$\sigma^a * \sigma^b - \sigma^b * \sigma^a = i\theta^{ab} . \tag{3}$$

The equations of motion of the worldsheet fields are

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu = \partial_+\psi_-^\mu = \partial_-\psi_+^\mu = 0 , \tag{4}$$

where $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. Therefore, ψ_-^μ and ψ_+^μ are the right moving and the left moving components of ψ^μ. These equations are the same that appear for the superstring with ordinary worldsheet. For the next purposes we need the solutions of these equations. For closed string there is

$$X^\mu(\sigma, \tau) = x^\mu + 2\alpha'p^\mu\tau + 2L^\mu\sigma + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)}\right) , \tag{5}$$

where L^μ is zero if the direction X^μ is non-compact, and is $N^\mu R^\mu$ if this direction is compact on a circle with radius R^μ. In this case the momentum of the closed string is quantized i.e.,

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$p^\mu = \frac{M^\mu}{R^\mu}$. The integers N^μ and M^μ are the winding and the momentum numbers of the closed string respectively.

For open string we have the solution

$$X^\mu(\sigma,\tau) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma \ . \tag{6}$$

Note that this solution satisfies the boundary conditions of the variation of the action (1). In this variation there are infinite number of boundary terms. These terms contain derivatives that originate from the noncommutativity of the string worldsheet. The boundary condition

$$(\partial_\sigma X^\mu)_{\sigma_0} = 0 \ , \tag{7}$$

is sufficient to drop all the boundary terms of the above variation. $\sigma_0 = 0, \pi$ shows the boundaries of the open string worldsheet.

Consider global worldsheet supersymmetry transformations

$$\delta X^\mu = \bar{\epsilon}\psi^\mu \ ,$$
$$\delta\psi^\mu = -i\rho^a \partial_a X^\mu \epsilon \ . \tag{8}$$

Invariance of the action (1) under these transformations leads to the worldsheet super-current

$$J_a = \frac{1}{2}\rho^b \rho_a \psi^\mu * \partial_b X_\mu \ . \tag{9}$$

According to the equations of motion, this is a conserved current i.e., $\partial^a J_a = 0$.

3　Spacetime noncommutativity

Noncommutativity of the string worldsheet directly leads to the noncommutativity of the spacetime. Making use of the formula

$$e^{ip(\tau+k\sigma)} * e^{iq(\tau+l\sigma)} = e^{\frac{i}{2}\theta pq(k-l)} e^{ip(\tau+k\sigma)} e^{iq(\tau+l\sigma)} \ , \tag{10}$$

we obtain the following noncommutativities for the spacetime

$$[X^\mu(\sigma,\tau) \ , \ X^\nu(\sigma,\tau)]_* = \theta(2\alpha')^{3/2} \sum_{n\neq 0} (p^\mu \alpha_n^\nu - p^\nu \alpha_n^\mu) e^{-in\tau} \sin n\sigma$$

$$-2\alpha' \sum_{m\neq 0} \sum_{n\neq 0} \left(\frac{1}{mn}(\alpha_n^\mu \alpha_m^\nu - \alpha_n^\nu \alpha_m^\mu) e^{-i(m+n)\tau} \cos m(\sigma + \frac{1}{2}n\theta) \cos n(\sigma - \frac{1}{2}m\theta) \right) , \tag{11}$$

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from the open string point of view, and

$$[X^\mu(\sigma,\tau) \ , \ X^\nu(\sigma,\tau)]_* = 4i\theta\alpha'(p^\mu L^\nu - p^\nu L^\mu)$$
$$+i\theta(2\alpha')^{3/2}\sum_{n\neq 0}\left((\alpha_n^\mu p^\nu - \alpha_n^\nu p^\mu)e^{-2in(\tau-\sigma)} - (\tilde{\alpha}_n^\mu p^\nu - \tilde{\alpha}_n^\nu p^\mu)e^{-2in(\tau+\sigma)}\right)$$
$$+2i\theta\sqrt{2\alpha'}\sum_{n\neq 0}\left((\alpha_n^\mu L^\nu - \alpha_n^\nu L^\mu)e^{-2in(\tau-\sigma)} + (\tilde{\alpha}_n^\mu L^\nu - \tilde{\alpha}_n^\nu L^\mu)e^{-2in(\tau+\sigma)}\right)$$
$$-i\alpha'\sum_{n\neq 0}\sum_{m\neq 0}\left(\frac{1}{mn}(\tilde{\alpha}_n^\mu\alpha_m^\nu - \tilde{\alpha}_n^\nu\alpha_m^\mu)\sin(4mn\theta)e^{-2i(m+n)\tau}e^{2i(m-n)\sigma}\right) \ , \tag{12}$$

from the closed string point of view. The right hand sides of the relations (11) and (12) are non-zero. This is because of the non-zero value of the parameter θ. Therefore, the noncommutativity of the spacetime, resulted from the noncommutativity of the string worldsheet, depends on the fact that the propagating string in the spacetime is open or is closed.

Now consider the following expectation values of the commutators (11) and (12)

$$\langle 0|[X^\mu(\sigma,\tau) \ , \ X^\nu(\sigma,\tau)]_*|0\rangle = 0 \ , \qquad \text{for open string,}$$
$$\langle v|[X^\mu(\sigma,\tau) \ , \ X^\nu(\sigma,\tau)]_*|v\rangle = i\Theta^{\mu\nu}, \quad \text{for closed string} \ , \tag{13}$$

where $|v\rangle = |0,\tilde{0} \ ; \{M^\mu\}, \{N^\nu\}\rangle$ is a closed string state with the momentum numbers $\{M^\mu\}$ and the winding numbers $\{N^\mu\}$, and

$$\Theta^{\mu\nu} \equiv 4\alpha'\theta(p^\mu L^\nu - p^\nu L^\mu)$$
$$= 4\alpha'\theta\left(M^\mu N^\nu \frac{R^\nu}{R^\mu} - M^\nu N^\mu \frac{R^\mu}{R^\nu}\right) \ , \tag{14}$$

therefore, if some of the directions of the spacetime are compactified on tori with radii $\{R^\mu\}$, a closed string with noncommutative worldsheet, momentum numbers $\{M^\mu\}$ and winding numbers $\{N^\mu\}$, probes the expectation value of the noncommutativity of the spacetime like the relation (14). Open strings with noncommutative worldsheet do not probe the vacuum expectation value of the spacetime noncommutativity.

4   Poincaré symmetry

Poincaré transformations $\delta X^\mu = a^\mu_{\ \nu}X^\nu + b^\mu$ and $\delta\psi^\mu = a^\mu_{\ \nu}\psi^\nu$, are global symmetries of the superstring theory with ordinary worldsheet, where $a_{\mu\nu}$ is a constant antisymmetric tensor and b^μ is a constant vector. Two conserved currents are associated to them. For the superstring theory with noncommutative worldsheet, these transformations can be generalized.

Therefore, the effects of this generalization also appear in the currents. The generalized transformations are

$$\delta X^\mu = a^\mu_{\ \nu} X^\nu + b^\mu ,$$

$$\delta \psi^\mu = a^\mu_{\ \nu} (\psi^\nu + \psi^\nu * \phi - \psi^\nu \phi) , \tag{15}$$

where $\phi(\sigma, \tau)$ is a scalar of the worldsheet. It will be determined in terms of the coordinates σ and τ.

Making use of the equations of motion, the variation of the action (1) under the transformations (15), is

$$\delta S_* = \frac{i}{4\pi\alpha'} a_{\mu\nu} \int d^2\sigma \left(\bar{\psi}^\mu * \rho^a (\psi^\nu * \partial_a \phi - \psi^\nu \partial_a \phi) \right) . \tag{16}$$

For vanishing of this variation, one possibility is that $\partial_a \phi$ be constant i.e., independent of the coordinates σ and τ,

$$\partial_a \phi(\sigma, \tau) = c_a . \tag{17}$$

This equation has the solution

$$\phi(\sigma, \tau) = c_a \sigma^a + \phi_0 , \tag{18}$$

where c_0, c_1 and ϕ_0 are constants.

The currents associated to the transformations (15) are

$$P_a^\mu = \frac{1}{2\pi\alpha'} \, \partial_a X^\mu , \tag{19}$$

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\begin{aligned}
J_a^{\mu\nu} = & \frac{1}{4\pi\alpha'} \Big((X^\mu * \partial_a X^\nu - X^\nu * \partial_a X^\mu + \partial_a X^\nu * X^\mu - \partial_a X^\mu * X^\nu) \\
& + i\bar{\psi}^\mu * \rho_a (\psi^\nu * \phi - \psi^\nu \phi) - i\bar{\psi}^\nu * \rho_a (\psi^\mu * \phi - \psi^\mu \phi) \Big) \\
& + i (\bar{\psi}^\mu * \rho_a \psi^\nu - \bar{\psi}^\nu * \rho_a \psi^\mu) .
\end{aligned}
\tag{20}
$$

The constant ϕ_0 has no effects on the transformations (15) and on the current (20). According to the equations of motion, these currents are conserved i.e., $\partial^a P_a^\mu = 0$ and $\partial^a J_a^{\mu\nu} = 0$. If the noncommutative worldsheet changes to the ordinary worldsheet, the transformations (15) and the current (20) reduce to the ordinary case, as expected.

more generalization

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Transformations (15) can be generalized as

$$\delta X^\mu = a^\mu_{\ \nu} X^\nu + b^\mu \ ,$$

$$\delta \psi^\mu = a^\mu_{\ \nu}(\psi^\nu + b_1 \psi_1^\nu + b_2 \psi_2^\nu + ... + b_N \psi_N^\nu) \ , \tag{21}$$

where N is an integer and $\{b_1, b_2, ..., b_N\}$ are arbitrary coefficients, and

$$\psi_n^\mu = \psi_{n-1}^\mu * \phi - \psi_{n-1}^\mu \phi \quad , \quad 1 \le n \le N \ ,$$

$$\psi_0^\mu = \psi^\mu \ . \tag{22}$$

Again with the choice (18) for ϕ, the variation of the action (1) under the transformations (21) vanishes. Define differential operator D as

$$D = -\frac{1}{2} i \theta^{ab} c_a \partial_b = \frac{1}{2} i \theta (c_1 \partial_\tau - c_0 \partial_\sigma) \ , \tag{23}$$

therefore, ψ_n^μ can be written as

$$\psi_n^\mu = D^n \psi^\mu \ . \tag{24}$$

This simplifies the second transformation in (21) as the following

$$\delta \psi^\mu = a^\mu_{\ \nu} \sum_{n=0}^{N} b_n D^n \psi^\nu \quad , \quad b_0 = 1 \ . \tag{25}$$

For the special choices $b_n = \frac{1}{n!}$ and $N \to \infty$ this transformation becomes

$$\begin{aligned}
\delta \psi^\mu &= a^\mu_{\ \nu} \exp\left(\frac{1}{2} i \theta (c_1 \partial_\tau - c_0 \partial_\sigma)\right) \psi^\nu(\sigma, \tau) \\
&= a^\mu_{\ \nu} \psi^\nu\left(\sigma - \frac{i\theta}{2} c_0 \ , \ \tau + \frac{i\theta}{2} c_1\right) \ ,
\end{aligned} \tag{26}$$

which follows by combination of the shifts on the worldsheet coordinates and a rotation in the spacetime.

The currents associated to the transformations (21), are the current (19) and

$$\begin{aligned}
J_a^{\mu\nu} &= \frac{1}{4\pi\alpha'} \Big((X^\mu * \partial_a X^\nu - X^\nu * \partial_a X^\mu + \partial_a X^\nu * X^\mu - \partial_a X^\mu * X^\nu) \\
&\quad + i \sum_{n=0}^{N} b_n (\bar{\psi}^\mu * \rho_a D^n \psi^\nu - \bar{\psi}^\nu * \rho_a D^n \psi^\mu) \Big) \ ,
\end{aligned} \tag{27}$$

which is conserved i.e., $\partial^a J_a^{\mu\nu} = 0$. This is more general than the current (20).

Note that the parameters $\{b_n\}$, c_0 and c_1 in the transformation (25) and in the current (27), remain arbitrary. For $c_0 = \pm c_1$, the operator D is proportional to $\partial_\mp$. In this case,

according to the equation (10), the effects of the noncommutativity of the worldsheet on the fermionic part of the current (27), for $n \geq 1$, are collected only in the derivative D. That is, the star product appears as usual product. Also the transformation (25) for $n \geq 1$ only has derivatives of the left moving or the right moving components of the worldsheet fermions $\{\psi^\mu\}$.

5   The fields $g_{\mu\nu}$, $B_{\mu\nu}$ and Φ

We are interested in to know the effects of the noncommutativity of the string worldsheet on the metric, antisymmetric tensor and dilaton. We discuss on these fields both in the bosonic string and in the superstring theories. The states of these fields can be extracted from their vertex operators.

For the bosonic string consider the operator

$$\Omega^{\mu\nu}(p) = -\frac{2i}{\pi\alpha'} \int d^2\sigma : \partial_- X^\mu * \partial_+ X^\nu * e^{ip.X} : \quad . \tag{28}$$

Making use of the solution (5), therefore from the state

$$\Omega^{\mu\nu}(0)|0 , \tilde{0} ; p = 0\rangle , \tag{29}$$

we read the state

$$e^{-4i\theta} \alpha^\mu_{-1} \tilde{\alpha}^\nu_{-1}|0 , \tilde{0} ; p = 0\rangle . \tag{30}$$

According to this state we have

$$g^{\mu\nu}_\theta = e^{-4i\theta} g^{\mu\nu} ,$$
$$B^{\mu\nu}_\theta = e^{-4i\theta} B^{\mu\nu} ,$$
$$\Phi_\theta = e^{-4i\theta} \Phi . \tag{31}$$

Therefore, these fields take only a phase. Modification of the dilaton changes the string coupling constant.

For the superstring, $g_{\mu\nu}$, $B_{\mu\nu}$ and Φ are the NS⊗NS sector fields. The states of these fields can be extracted from the following state

$$-\frac{2i}{\pi} \int d^2\sigma : \psi^\mu_- * \psi^\nu_+ : |0 , \tilde{0} ; p = 0\rangle . \tag{32}$$

From this state we read the state

$$e^{-i\theta} b^\mu_{-1/2} \tilde{b}^\nu_{-1/2}|0 , \tilde{0} ; p = 0\rangle . \tag{33}$$

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According to this state, we obtain

$$g_\theta^{\mu\nu} = e^{-i\theta} g^{\mu\nu} \ ,$$
$$B_\theta^{\mu\nu} = e^{-i\theta} B^{\mu\nu} \ ,$$
$$\Phi_\theta = e^{-i\theta} \Phi \ . \tag{34}$$

Therefore, the corrections of the metric, antisymmetric tensor and dilaton in the superstring theory are different from their corrections in the bosonic string theory. Since θ-parameter is real, the real parts of these fields can be interpreted as physical fields.

6   Conclusions

The noncommutativity of the string worldsheet leads to the noncommutativity of the spacetime. The latter depends on probing by open or closed string. If some of the spacetime directions are compactified on tori, the noncommutativity of the spacetime depends on the momentum numbers and the winding numbers of the probing closed string.

By adding some additional terms to the Poincaré transformations of the worldsheet fermions, the Poincaré symmetry of the superstring was generalized. The noncommutativity of the superstring worldsheet permits this generalization and consequently the generalized form of the associated conserved current.

The NS$\otimes$NS fields of the type II superstring (i.e., the metric, antisymmetric tensor and dilaton) changed only by a phase. The changes of these fields in the bosonic string theory, are different from their changes in the superstring theory.

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