SL(3,C)-GROUP ELEMENT SOLUTION OF THE PRINCIPAL CHIRAL FIELD PROBLEM

M.A. MUKHTAROV
Institute of Mathematics and Mechanics
370602 Baku, F. Agaev str.9, A erbaijan

The group element solutions of the principal chiral field problem are constructed by means of discrete symmetry transformations for the algebra SL(3,C).

1. The problem of constructing the exact solutions of nonlinear evolution equations in the explicit form remains important for the present time. Among these so called integrable system the four dimensional self-dual Yang Mills (SDYM) equations plays the central role being the universal integrable system from which the systems in lower than four dimensions can be obtained by symmetry reduction or by imposing constrains on Yang Mills potentials. The problem of integration of SDYM has successfully solved only for the case of SL(2,C) algebra and for instanton number not greater than two. The famous ADHM ansatz [1] gives the qualitative description of instanton solution but not its explicit form. Two effective methods of integration of SDYM for arbitrary semisimple algebra have been proposed in series of papers [2]. Another, the discrete symmetry transformation approach has been suggested [3] that allows to generate new solutions from the old ones. This method has been applied to many cases, for instance, the exact solutions of principal chiral field problem were obtained in [4] for the case of SL(2,C) algebra and in [5] for SL(3,C) and the rest semisimple algebras of the rank greater than two.

This work must be considered as a continuation of the paper [5]. The purpose of the present paper is to do the same for the group-valued element what is important for applications.

2. Let us remind the basic statements from [5].

Equations of the principal chiral field problem are the systems of equations for the element f, taking values in the semisimple algebra,

$$\left(\theta_i-\theta_j\right)\frac{\partial^2 f}{\partial x_i\partial x_j}=\left[\frac{\partial f}{\partial x_i},\frac{\partial f}{\partial x_j}\right] , \tag{1}$$

In the case of two-dimensional space: $\theta_1=1,\ \theta_2=-1,\ x_1=\xi,\ x_2=\nu$.

For the case of a semisimple Lie algebra and for an element f being a solution of (1), the following statement takes place:

There exists such an element S taking values in a gauge group that

$$S^{-1}\frac{\partial S}{\partial x_i}=\frac{1}{\tilde f_-}\left[\frac{\partial \tilde f}{\partial x_i},X_M^+\right]-\theta_i\frac{\partial}{\partial x_i}\frac{1}{\tilde f_-}X_M^+ \tag{2}$$

Here X_M^+ is the element of the algebra corresponding to its maximal root divided by its norm, i.e.,

$$\left[X_M^+,\ X^-\right]=H,\ \left[H,\ X^\pm\right]=\pm 2X^\pm ,$$

$-\ \tilde f_-$ - is the coefficient function in the decomposition of $\tilde f$ of the element corresponding to the minimal root of the algebra, $\tilde f=\sigma f\sigma^{-1}$ and where σ is an automorfism of the algebra, changing the positive and negative roots.

In the case of algebra SL(3,C) we'll consider the case of three dimensional representation of algebra and the following form of $\sigma=\begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix}$.

The discrete symmetry transformation, producing new solutions from the known ones, is as follows:

$$\frac{\partial F}{\partial x_i}=S\frac{\partial \tilde f}{\partial x_i}S^{-1}+\theta_i\frac{\partial S}{\partial x_i}S^{-1} \tag{3}$$

Using the equations of the principal chiral field problem for the group-valued element

$$\theta_i(g)_{x_i} g^{-1} = (f)_{x_i}$$

the relations (3) can be rewritten as

$$\theta_i(S_n\sigma g_n)_{x_i}(S_n\sigma g_n)^{-1} = (f_{n+1})_{x_i} \quad ,$$

So we see that the group valued elements g_{n+1} and g_n are connected by the relation

$$g_{n+1} = S_n\sigma g_n \tag{6}$$

3. Let's represent the explicit formulae of the recurrent procedure of obtaining the group-valued element solutions of the self-duality equations in the case of SL(3,C) algebra .

At every step, as it shown in [5], S is upper triangular matrix and can be represented in the following form:

$$S_n = exp(\beta_1)_n X_1^+ \; exp(\beta_{1,2})_n X_{1,2}^+ \; exp(\beta_2)_n X_2^+ \; exp(\beta_0)_n H \quad , \tag{7}$$

where $H=h_1+h_2$ and for g_n we use the following parameterization:

$$\begin{aligned} g_n = {} & exp(\eta_1^+)_n X_1^+ \; exp(\eta_{1,2}^+)_n X_{1,2}^+ \; exp(\eta_2^+)_n X_2^+ \; exp((t_1)_n h_1 + (t_2)_n h_2) \times \\ & \times exp(\eta_2^-)_n X_2^- \; exp(\eta_{1,2}^-)_n X_{1,2}^- \; exp(\eta_1^-)_n X_1^- \end{aligned} \tag{8}$$

with

$$g_0 = exp(\eta_1^+)_0 X_1^+ \; exp(\eta_{1,2}^+)_0 X_{1,2}^+ \; exp(\eta_2^+)_0 X_2^+ \; exp((t_1)_0 h_1 + (t_2)_0 h_2)$$

as an initial solution.

Hereafter, $X_1^\pm, X_2^\pm, X_{1,2}^\pm, h_1, h_2$ are the generators of SL(3,C) algebra.

Following the general scheme from [5] and using (2) we have at

(0)-step:

$$(t_i)_0 = \tau_i^{-1} \equiv \nu_i \;, (\eta_i^+)_0 = \alpha_i^{-1}, i = 1,2 \;, (\eta_{1,2}^+)_0 = \alpha_{1,2}^{-1,0} \quad ;$$

(1)-step:

$$(t_1)_1 = -\nu_1 + ln\left(-\frac{\alpha_{1,2}^{-1,0}}{\alpha_{1,2}^{0,0}}\right) \;, \; (t_2)_1 = -\nu_2 + ln\left(-\frac{\alpha_{1,2}^{0,-1}}{\alpha_{1,2}^{0,0}}\right) \;,$$

$$(\eta_1^-)_1 = -\frac{\alpha_2^{-1}}{\alpha_{1,2}^{-1,0}} exp\,\delta_1 \;, \; (\eta_2^-)_1 = \frac{\alpha_1^{-1}}{\alpha_{1,2}^{0,-1}} exp\,\delta_2 \;, (\eta_{1,2}^-)_1 = \frac{1}{\alpha_{1,2}^{-1,0}} exp(\delta_1 + \delta_2),$$

$$(\eta_1^+)_1 = -\frac{det\begin{pmatrix}\alpha_1^{-1} & \alpha_1^0 \\ \alpha_{1,2}^{-1,0} & \alpha_{1,2}^{0,0}\end{pmatrix}}{\alpha_{1,2}^{-1,0}} \;, (\eta_2^+)_1 = -\frac{det\begin{pmatrix}\alpha_2^{-1} & \alpha_2^0 \\ \alpha_{1,2}^{0,-1} & \alpha_{1,2}^{0,0}\end{pmatrix}}{\alpha_{1,2}^{0,-1}} \;, (\eta_{1,2}^+)_1 = \frac{det\begin{pmatrix}\alpha_{1,2}^{-1,0} & \alpha_{1,2}^{-1,1} \\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1}\end{pmatrix}}{\alpha_{1,2}^{-1,0}} \;,$$

$$\delta_i = 2\nu_i - \nu_j \;, i \neq j \quad ;$$

(2)-step:

$$\left(\eta_1^-\right)_2 = -\frac{\det\begin{pmatrix}\alpha_2^{-1} & \alpha_2^{0}\\ \alpha_{1,2}^{1,-1} & \alpha_{1,2}^{1,0}\end{pmatrix}}{\det\begin{pmatrix}\alpha_{1,2}^{-1,0} & \alpha_{1,2}^{-1,1}\\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1}\end{pmatrix}}\exp\delta_1 \;, \quad \left(\eta_2^-\right)_2 = \frac{\det\begin{pmatrix}\alpha_1^{-1} & \alpha_1^{0}\\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{1,0}\end{pmatrix}}{\det\begin{pmatrix}\alpha_{1,2}^{0,-1} & \alpha_{1,2}^{0,0}\\ \alpha_{1,2}^{1,-1} & \alpha_{1,2}^{1,0}\end{pmatrix}}\exp\delta_2 \;,$$

$$\left(\eta_{1,2}^-\right)_2 = \frac{1}{\det\begin{pmatrix}\alpha_{1,2}^{-1,0} & \alpha_{1,2}^{-1,1}\\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1}\end{pmatrix}}\exp\left(\delta_1+\delta_2\right),$$

$$\left(\eta_1^+\right)_2 = \frac{\det\begin{pmatrix}\alpha_1^{-1} & \alpha_1^{0} & \alpha_1^{1}\\ \alpha_{1,2}^{-1,0} & \alpha_{1,2}^{0,0} & \alpha_{1,2}^{1,0}\\ \alpha_{1,2}^{-1,1} & \alpha_{1,2}^{0,1} & \alpha_{1,2}^{1,1}\end{pmatrix}}{\det\begin{pmatrix}\alpha_{1,2}^{-1,0} & \alpha_{1,2}^{-1,1}\\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1}\end{pmatrix}_1} \;, \quad \left(\eta_2^+\right)_2 = \frac{\det\begin{pmatrix}\alpha_2^{-1} & \alpha_2^{0} & \alpha_2^{1}\\ \alpha_{1,2}^{0,-1} & \alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1}\\ \alpha_{1,2}^{1,-1} & \alpha_{1,2}^{1,0} & \alpha_{1,2}^{1,1}\end{pmatrix}}{\det\begin{pmatrix}\alpha_{1,2}^{0,-1} & \alpha_{1,2}^{0,0}\\ \alpha_{1,2}^{1,-1} & \alpha_{1,2}^{1,0}\end{pmatrix}_2} \;,$$

$$\left(\eta_{1,2}^+\right)_2 = \frac{\det\begin{pmatrix}\alpha_{1,2}^{-1,0} & \alpha_{1,2}^{-1,1} & \alpha_{1,2}^{-1,2}\\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1} & \alpha_{1,2}^{0,2}\\ \alpha_{1,2}^{1,0} & \alpha_{1,2}^{1,1} & \alpha_{1,2}^{1,2}\end{pmatrix}}{\det\begin{pmatrix}\alpha_{1,2}^{-1,0} & \alpha_{1,2}^{-1,1}\\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1}\end{pmatrix}}$$

Here, α_1^{i}, α_2^{j}, $\alpha_{1,2}^{i,j}$ - chains of solutions of principle chiral field problem determined by formulae (9-12) from [5].

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M.A. Mıxtarov

SL(3,C)-QRUP ELEMENTİ ÜÇÜN ƏSAS KİRAL SAHƏNİN MƏSƏLƏSİNİN HƏLLİ

SL(3,C)- cəbri halında diskret spektr metodu vəsitəsilə əsas kiral sahənin məsələsinin həlli tapılmışdır.

М.А. Мухтаров

РЕШЕНИЕ ЗАДАЧИ ГЛАВНОГО КИРАЛЬНОГО ПОЛЯ ДЛЯ SL(3,C)-ГРУППОВОГО ЭЛЕМЕНТА

Построены решения для группового элемента уравнений главного кирального поля методом дискретных симметрий в случае алгебры SL(3,C).

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