CLASSICAL FACTORIZATION METHOD FOR THE NON-STATIONARY SYSTEM

R.G. AGAYEVA
Institute of Physics of the National Academy of Sciences of Azerbaijan
370143, Baku, H.Javid ave.33.

The classical factorization method is constructed for the non-stationary system with the use of quantum integrals of motion.

The classical factorization method (CFM) developed by Schrödinger [l] and extended by Infeld and his collaborators [2] allows the eigenfunctions (EF) and eigenvalues (EV) to be constructed for the stationary problems.

Within the framework of CFM the Hamiltonian for the harmonic oscillator is known to be represented as $\hbar\omega\left(\hat{a}^+\hat{a}^- + \frac{1}{2}\right)$, where ω is a frequency, $\hat{a}^+$, $\hat{a}^-$ are the Bose rising and lowering operators, respectively. Then the EV_s and the EF_s of the Hamiltonian are defined by the algebraic way provided the energy EV have a lower limit.

To solve the non-stationary problem means to determine the wavefunctlon ψ. satisfying the wave equation $i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$, where $\hat{H}$ is the Hamiltonian of the problem under consideration. However, the wave function of the non-stationary problem is not EF of $\hat{H}$ and, there fore, it is impossible for the CFM to be extended to the non-stationary case directly.

The wave function of the non-stationary system might be determined if this wave function obeys not only the wave equation but simultaneously is the EF of a certain operator

$$\hat{K} = \hat{A}^+\hat{A}^- + \frac{1}{2} \tag{1}$$

where $\hat{A}^{\pm}$ are the Bose rising and lowering operators for the given non-stationary system and the EV of $\hat{K}$ have the lower limit. Such a situation is realized provided

$$\left[\hat{K}, i\hbar\frac{\partial}{\partial t} - \hat{H}\right] = 0 \tag{2}$$

i.e. only on condition that $\hat{K}$ is the quantum integral of motion.

The aim of the present work is to show, with a harmonic oscillator with a time-dependent frequency being used as an example, that the CFM may be developed for the non-stationary system provided the method of the quantum motion integrals is used.

There is a further point to be made, in the case of the non-stationary problem one can solve the EV problem for the operator $\hat{K}$ instead of the corresponding wave equation. In the stationary case this operator transform to the energy operator.

Let us consider a non-stationary harmonic oscillator described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2(t)\hat{x}^2}{2} \tag{3}$$

where x is a usual canonical coordinate, p is its conjugate momentum, and m is a mass.

It is known that lowering and rising operators for such system [3] are:

$$\hat{A}^- = \frac{i}{\sqrt{2\hbar}}\left(\frac{\varepsilon\hat{p}}{\sqrt{m}} - \dot{\varepsilon}\sqrt{m}\hat{x}\right),$$
$$\hat{A}^+ = -\frac{i}{\sqrt{2\hbar}}\left(\frac{\varepsilon^*\hat{p}}{\sqrt{m}} - \dot{\varepsilon}^*\sqrt{m}\hat{x}\right) \tag{4}$$

where the function $\varepsilon(t)$ is a definite solution of the classical harmonic oscillator equation

$$\ddot{\varepsilon} + \omega^2(t)\varepsilon = 0 . \tag{5}$$

The following commutation relationship holds

$$\left[\hat{A}^-, \hat{A}^+\right] = 1 \tag{6}$$

It is easy to check that

$$\left[i\hbar\frac{\partial}{\partial t} - \hat{H}, \hat{A}^{\pm}\right] = 0 \tag{7}$$

i.e. the operators (4) are invariants.

Formulae (5) and (6) give the following equality,

$$\dot{\varepsilon}\varepsilon^* - \dot{\varepsilon}^*\varepsilon = 2i \tag{8}$$

which is valid for any moment of the t .

Let us introduce an operator $\hat{K}$ according to (1) where $\hat{A}^{\pm}$ are given by the expressions (4). If $\hat{H}$ has the form (3) it is easy in compliance with (2) to be convinced that $\hat{K}$ is the motion integral. This means that $\hat{K}$ commutes with the operator $\left(i\hbar\frac{\partial}{\partial t} - \hat{H}\right)$. Hence, these operators have exactly the same set of EF_s. Consequently, the wave function of non-stationary harmonic oscillator can if be found if the EV problem for operator $\hat{K}$ is solved:

$$\hat{K}\psi = k\psi \tag{9}$$

Construct the following motion integrals

$$\hat{X}_0 = \left(\hat{A}^- + \hat{A}^+\right)/\sqrt{2} \tag{10}$$

and

$$\hat{P}_0 = \left(\hat{A}^- - \hat{A}^+\right)/i\sqrt{2} \tag{11}$$

They are referred to as the operator of the initial coordinate and the operator of initial impulse, respectively [4]. Let us express $\hat{K}$ from (l) in terms of these operators:

$$\hat{K} = \left(\hat{X}_0^2 + \hat{P}_0^2\right)/\sqrt{2} \tag{12}$$

$\hat{X}_0$ and $\hat{P}_0$ are the Hermitian operators. Then these operators have the real EV that places the lower limit of EV of the expression (12): $k \geq 0$ This enables to apply CFM to solving the problem (9).

Denote the quantities belonged to the ground state, i. e. to the lowest EV of the operator $\hat{K}$, by subscript "0".Then

$$\hat{K}\psi_0 = k_0\psi_0 \tag{13}$$

Let us multiply the equation (13) on the left-hand side by and make use of the commutation relationship (6) taking into account (1). Finally, instead of equation (13) we obtain $\hat{K}\hat{A}^-\psi_0 = (k_0 - 1)\psi_0$. Since k_0 is the lowest EV of the operator $\hat{K}$ it follows that

$$\hat{A}^-\psi_0 = 0 \tag{14}$$

whence

$$\psi_0 = C_0 exp\left(im\dot{\varepsilon}x^2 / 2\hbar\varepsilon\right) \tag{15}$$

and

$$C_0 = \sqrt[4]{\frac{m}{\pi\hbar\varepsilon^2}} \tag{16}$$

is calculated from the normalization for ψ_0 . By means of (14) we get from the expression (13) $k_0 = 1/2$.

Let us multiply the equality (13) on the left-hand side by $\hat{A}^+$ and make use of the expressions (6) and (1). We obtain $\hat{K}\hat{A}^+\psi_0 = (k_0 + 1)\hat{A}^+\psi_0$, whence $\psi_1 = C_1\hat{A}^+\psi_0$, $k_1 = k_0 + 1$. Using the mathematical induction method one can prove that $\psi_n = C_1\hat{A}^+\psi_{n-1}$, $k_n = n + \frac{1}{2}$. The value of C_n is given by the normalization conduction. On the whole we get the wave function of the non-stationary harmonic oscillator in the form

$$\psi_n = (n!)^{-1/2}\left(\hat{A}^+\right)^n\psi \tag{17}$$

that exactly coincides with the well known result for the system under consideration [4].

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R.Q. Ağayeva

QEYRİ STASİONAR SİSTEMLƏR ÜÇÜN KLASSİK FAKTORİZASİYA METODU

Hərəkətin kvant inteqrallarının köməyi ilə qeyri stasionar sistemlər üçün klassik faktorizasiya metodu işlənib.

Р.Г.Агаева

КЛАССИЧЕСКИЙ МЕТОД ФАКТОРИЗАЦИИ ДЛЯ НЕСТАЦИОНАРНЫХ СИСТЕМ

С помощью квантовых интегралов движения развит классический метод факторизации для нестационарных систем.

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