REALIZATION OF THE TOMOGRAPHIC PRINCIPLE IN QUANTUM STATE OF DAMPED OSCILLATOR

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The general principle for the tomographic approach to quantum state reconstruction, which until new has been based on a simple rotation transformation in the phase space is considered. The realization of the principle in specific example is presented.

1. Introduction

In 1932 Wigner [1] introduced a real function $W(q,p)$ which is related by Fourier transform with complex density matrix $\rho(x,x')$. The Wigner function has the specific properties which are similar to properties of a probability distribution function of classical statistical mechanics. The motivation to introduce such function was to make the description of quantum state closer to intuitively more familiar description of classical state by means of probability distribution on the phase space. Moyal [2] has formulated evolution equation of quantum state in terms of Wigner function. The Moyal formulation of quantum mechanics showed very clearly what is similarities and differences of the classical and quantum fluctuations.

Nevertheless the Wigner function can not be considered as joint probability distribution on phase space. The obvious reason for this is the fact that the Wigner function can take negative values for quantum states [3-5]. The Wigner function is used to study the evolution of quantum systems [5-8] since it provides a convenient representation similar to classical picture of the evolution.

Recently, in [9-11] the probability representation of quantum mechanics was introduced and the new evolution equation was derived, which was a generalization of the result obtained in [12], where the role of the Wigner function was played by the particles position in an ensemble of rotated and scaled reference frames in the system's classical phase space (the classical representation of quantum mechanics uses the symplectic tomography procedure suggested for measuring quantum states [13,14]. Tomography is well known in the field of medicine where it is extensively used for image reconstruction in diagnostic systems. It is based on the possibility of recording transmission profiles of the radiation which has penetrated a living body from various directions. In quantum optics, one has the opportunity of measuring all possible, so that tomography can be easily implemented. In fact Vogel and Risken [ ] pointed out that the marginal distribution is just the Rodon transform (or "tomography") of the Wigner function.

By inverting the Radon transform, one can obtain the Wigner function and then recover the state, this is the basis of the method proposed by Smithey et al [15].

The aim of this paper is to consider the tomographic principle and investigate in a frame of this principle the quantum system described by the quadratic non-stationary Hamiltonian.

2. Symplectic tomography

In the usual optic homodyne tomography the observed quantities are the quadratures $\hat{x}_\varphi = \hat{q}cos\varphi + \hat{p}sin\varphi$ obtained as mixtures of position $\hat{q}$ and momentum $\hat{p}$ by means of a rotation g in phase space

$$\begin{bmatrix} q \\ p \end{bmatrix} \rightarrow g\begin{bmatrix} q \\ p \end{bmatrix}, g = \begin{bmatrix} cos\varphi & sin\,\varphi \\ -sin\varphi & cos\varphi \end{bmatrix} \tag{2.1}$$

The quadrature histograms $w(x,\varphi)$ also called marginal distributions are projections (Rodon transformation) of Wigner function [1]

$$w(x,\varphi) = \int W(qcos\varphi - psin\varphi, qsin\varphi + pcos\varphi)dp \tag{2.2}$$

On the other hand the resulting marginal distribution $w(x,\varphi)$ is [13]

$$w(x,\varphi) = < x_\varphi|\hat{\rho}|x\varphi> = < q|G(g)\hat{\rho}\, G^{-1}(g)|q>$$

where $|x_\varphi>$ are eigenkets of quadrature operators and $G(g)$ is the unitary group representation for the transformation g. In this case

$$G(g) = exp\left[i\varphi\left(\frac{\hat{p}^2}{2} + \frac{q^2}{2}\right)\right] \tag{2.4}$$

As was shown in [12] for the generic linear combination of the position q and momentum p, which a measurable observable in the phase space

$$\hat{x} = \mu\hat{q} + \nu\hat{p} + \delta \tag{2.5}$$

where μ, ν, δ are real parameters, the marginal distribution $\omega(x,\mu,\nu)$ is related to the state of the quantum system expressed in terms of its Wigner function $W(q,p)$ as follows:

$$\omega(x,\mu,\nu) = \int exp[-ik(x - \mu q - \nu p)W(q,p)\frac{dkdqdp}{(2\pi)^2} \tag{2.6}$$

where $x=X-\delta$. By means of the Fourier transform of the function ω, one can then obtain the relation

$$W(q,p) = (2\pi)^2 z^2 \tilde{\omega}(z,-zq,-zp) \tag{2.7}$$

where $-zq$, $-zp$ and z are the conjugate variables to μ, ν and x respectively and the Fourier transform $\widetilde{\omega}$ has the property

$$\omega\, z, -zq, -zp) = \frac{1}{z^2}\omega\,(1, -q, -p) \qquad (2.8)$$

It is worth remarking that in this case the connection between the Wigner function and the marginal distribution is simply guaranteed by means of the Fourier transform instead of the Rodon transform.

The procedure developed is called "symplectic tomography" [13], since in this case the marginal distribution is obtained by using a symplectic transformation g belonging to the symplectic group ISp(2,R)

$$\begin{bmatrix} q \\ p \end{bmatrix} \to g\begin{bmatrix} q \\ p \end{bmatrix}, g = \begin{bmatrix} cos\varphi & sin\,\varphi \\ -\,sin\varphi & cos\varphi \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \qquad (2.9)$$

For this transformation, one has

$$\mu = \lambda cos\varphi, \quad \nu = \lambda^{-1} sin\varphi, \delta = 0 \qquad (2.10)$$

This, for the realization of the scheme, the element g is the product of squeezing and rotation operators. This means that for our scheme the representation operator is

$$G(g) = exp\left[i\varphi\left(\frac{\hat{p}^2}{2} + \frac{q^2}{2}\right)\right]exp\left[\frac{i\lambda}{2}\left(\hat{q}\,\hat{p} + \hat{p}\,\hat{q}\right)\right] \qquad (2.11)$$

3. Marginal distribution for quantum dumped oscillator.

Let us consider a quantum system described by Hermitian non-stationary Hamiltonian [16]

$$\hat{H} = \frac{1}{2}\left(\hat{p}^2 e^{2\Gamma(t)} + \omega_0^e(t) e^{2\Gamma(t)}\hat{x}^2\right) - f(t)e^{2\Gamma(t)}\hat{x} \qquad (3.1)$$

The wave functions for the Fock states of this system ψ_n have the form

$$\psi_n(x,t) = (n!)^{-1/2}\left(\frac{\varepsilon^*}{2\varepsilon}\right)^{n/2}(\pi\varepsilon^2)^{-1/4}\, exp\left[\frac{i\dot{\varepsilon}}{2\varepsilon}e^{2\Gamma(t)}x^2 - \frac{x\delta}{\varepsilon} - \frac{\varepsilon^*}{4\varepsilon}\delta^* - \frac{1}{4}|\delta|2 - \right.$$
$$\left. -\frac{i}{2}\int Im(\delta\,\delta^*)\right]H_n\left[\frac{x + Re(\varepsilon^*\delta)}{\lfloor\varepsilon\rfloor}\right] \qquad (3.2)$$

where the $H_n(x)$ are Hermite polynomials and $\varepsilon(t)$ is a complex function satisfying the equation

$$\ddot{\varepsilon} + 2\dot{\Gamma}(t)\dot{\varepsilon} + \omega_0^2(t)\varepsilon = 0 \qquad (3.3)$$

and the additional relation

$$e^{2\Gamma(t)}(\varepsilon^*\varepsilon - \varepsilon\,\dot{\varepsilon}^* = 2i \qquad (3.4)$$

and

$$\delta(t) = -i\int\varepsilon(\tau)e^{2\Gamma(t)}f(t)d\tau$$

The corresponding Wigner function is as follows:

$$W_n(p,q) = 2(-1)^n e^{-2z(t)}Ln(4z(t) \qquad (3.5)$$

$$z(t) = \frac{1}{2}\left[|\varepsilon|^2 p^2 + |\dot{\varepsilon}|^2 e^{4\Gamma}q^2 - 2e^{2\Gamma}Re\dot{\varepsilon}\,\varepsilon^*)qp + I_m(\varepsilon^*\delta)p - e^{2\Gamma}Im(\varepsilon^*\delta)q + |\delta|^2 + ie^{2\Gamma}Re(\varepsilon\dot{\varepsilon}^*)\right.$$

The marginal distribution (2.6) as it was shown above is expressed in terms of its Wigner function. Then the marginal distribution $\omega_m(x, \mu, \nu)$ for the Fock states of our system is as follows

$$\omega_n(x, \mu, \nu, t) = \int exp\left[-ik(x - \mu q - \nu p)\right]W_n(q, p, t)\frac{d\,k\,d\,q\,d\,p}{(2\pi)^2} \qquad (3.6)$$

Substituting (3.5) in (3.6) and taking for simplicity $f(t)=0$ we obtained the exact expression for marginal distribution in the following form:

$$\omega_n(x, \mu, \nu, t) = \frac{1}{2^n n!\sqrt{\pi\varepsilon\varepsilon^*(a^2+b^2)}}exp\left(-\frac{x^2}{\varepsilon\varepsilon^*(a^2+b^2)}\right)H_n^2\left(\frac{x}{\sqrt{\varepsilon\,\varepsilon^*(a^2+b^2)}}\right) \qquad (3.7)$$

where

$$a=\frac{1}{\varepsilon\varepsilon^*}\cdot exp\left[\Gamma(t)\right]\nu\left(\varepsilon^*\varepsilon+\varepsilon\dot{\varepsilon}^*\right)+\mu\ ,$$

$$b=\nu/\varepsilon\varepsilon^*\ .$$

In the following paper we will obtain the smoothed Wigner function of our system and its smoothed marginal distribution and compare both expressions.

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SÖNMƏDƏ OLAN OSSİLYATORUN KVANT HALLARI ÜÇÜN TOMOQRAFİK PRİNSİPLƏRİN HƏYATA KEÇİRİLMƏSİ

Fırlanmanın faza fəzasında sadə çeviricisiyə əsaslanan kvant hallarının təsviri üçün tomoqrafik yanaşma sisteminin ümumi prinsipləri nəzərdən keçirilib. Bu prinsipin həyata keçirilməsi xüsusi misalda təqdim olunmuşdur.

Э.А. Ахундова

РЕАЛИЗАЦИЯ ТОМОГРАФИЧЕСКОГО ПРИНЦИПА ДЛЯ КВАНТОВЫХ СОСТОЯНИЙ ЗАТУХАЮЩЕГО ОСЦИЛЛЯТОРА

Рассмотрен общий принцип томографического подхода для описания квантовых состояний системы, который основан на простом преобразовании вращения в фазовом пространстве. Представлена реализация этого принципа на особом примере.

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