STATIONARY AXISYMETRIC GRAVITY AS A PRINCIPAL CHIRAL FIELD MODEL

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The stationary axisymmetric gravity equations (Ernst equations) are reduced to the principal chiral field problem with moving poles. Applying of the discrete symmetry transformations is discussed.

1. The problem of constructing the exact solutions of nonlinear evolution equations in the explicit form remains important for the present time. The existence of very rich integrable structure of Einstein equations have been conjectured by different authors. But the real discoveries of its integrability properties of stationary axisymmetric version of these equations, known as Ernst equations, and effective procedures for construction of solutions have been started in the papers of Belinskii and Zaharov [1]. In these papers the inverse scattering methods have been developed for vacuum gravitational fields. Among other approaches we have to point out the algebra-geometrical approach of Korotkin, Matveev and Nicolai [2-4]. Ernst equations as almost all so called integrable system can be obtained by symmetry reduction of the four dimensional self-dual Yang Mills (SDYM) equations that plays therefore the central role being the universal integrable system. The problem of integration of SDYM has successfully solved only for the case of SL(2,C) algebra and for instanton number not greater than two. The famous ADHM ansatz [5] gives the qualitative description of instanton solution but not its explicit form. Two effective methods of integration of SDYM for arbitrary semisimple algebra have been proposed in series of papers [6]. Another, the discrete symmetry transformation approach has been suggested [7] that allows to generate new solutions from the old ones. This method has been applied to many cases, for instance, the exact solutions of principal chiral field problem were obtained in [8] for the case of SL(2,C) algebra and in [9] for SL(3,C) and the rest semisimple algebras of the rank greater than two. This method can be effectively applied to the so called principle chiral problem with the moving poles [10].

This work shows how Ernst equations can be reduced to principle chiral problem with the moving poles with future possible application of the discrete symmetry transformation method.

2. The Ernst equation describing stationary axisymmetric metrics in general relativity can be represented in a form [11]:

$$(\rho g_z g^{-1})_{\bar z} + (\rho g_{\bar z} g^{-1})_z = 0 \quad , \tag{1}$$

where g is real and symmetric (2x2) matrix and $\det g = -\rho^2$, subscripts stands for partial derivatives throughout this paper.

Using the known formula from matrix theory

$$\mathrm{sp}(g_t g^{-1}) = \frac{\partial}{\partial t}\ln\det g \tag{2}$$

and taking trace from both sides of (1), we have:

$$(\rho\frac{\partial}{\partial z}(-\rho^2))_{\bar z} + (\rho\frac{\partial}{\partial \bar z}(-\rho^2))_z = 0$$

Finally, we get the D'Alambert equation for ρ

$$\rho_{z,\bar z} = 0$$

and its solution in terms of two arbitrary functions: $\rho = \varphi_1(z) + \varphi_2(\bar z)$.

Due to conformal invariance of the theory we can without loss of generality put $\varphi_1(z) \to z$, $\varphi_2(\bar z) \to \bar z$ and get the expression for ρ as:

$$\rho = z + \bar z \tag{3}$$

Then we can write currents of the corresponding linear system in a form:

$$\begin{aligned}(z+\bar z)g_z g^{-1} &= F_z\\ (z+\bar z)g_{\bar z} g^{-1} &= -F_{\bar z}\end{aligned} \tag{4}$$

If we take the complex conjugate of the first equation of (4) and compare the result with the second one then we come to the conclusion that F is pure imaginary function, i.e.

$$F^* = -F \tag{5}$$

Then using (2) we have

$$\mathrm{sp}F_z = \mathrm{sp}(\rho g_z g^{-1}) = \rho\frac{\partial}{\partial z}\ln\rho^2 = 2$$

Thus we have the second constrained property of F :

$$\begin{aligned}\mathrm{sp}F_z &= 2\\ \mathrm{sp}F_{\bar z} &= -2\end{aligned} \tag{6}$$

Let's introduce the element $\bar g = g\sigma$, where σ is a matrix of inner automorhpism having the form:

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

One can be convinced by the direct check in the following properties:

$$\begin{aligned} &\mathrm{sp}\bar{g} = 0 \\ &\bar{g}^2 = \rho^2 I \\ &\bar{g}^{-1} = \rho^{-2}\bar{g} \end{aligned} \qquad (7)$$

For the element θ defined as $\theta = F + \bar{g}$ we rewrite relation (4) as

$$\begin{aligned} \theta_z &= \bar{g}_z(\rho^{-1}\bar{g} + I) \\ \theta_{\bar{z}} &= \bar{g}_{\bar{z}}(-\rho^{-1}\bar{g} + I) \end{aligned} \qquad (8)$$

Note the evident relation coming directly from (7):

$$(\rho^{-1}\bar{g} + I)(-\rho^{-1}\bar{g} + I) = 0$$

From this relation and (8) it follows that

$$\begin{aligned} &\det\theta_z = \det\theta_{\bar{z}} \quad \text{or} \\ &\mathrm{rank}\theta_z = \mathrm{rank}\theta_{\bar{z}} = 1 \end{aligned} \qquad (9)$$

By changing variables

$$-\theta/4_{\bar{z}} \to f \quad , \quad -\bar{z} \to \bar{z} \qquad (10)$$

we eventually come to the equation

$$(z - \bar{z})f_{z,\bar{z}} = [f_{\bar{z}}, f_z] \qquad (11)$$

with additional relations

$$\mathrm{rank}f_z = \mathrm{rank}f_{\bar{z}} = 1 \qquad (11a)$$

$$\begin{aligned} \mathrm{sp}f_z &= 1/2 \\ \mathrm{sp}f_{\bar{z}} &= -1/2 \end{aligned} \qquad (11b)$$

$$\det \mathrm{Re}\,\theta = 1/16 \qquad (11c)$$

Equations (11) are equations of principle chiral field problem with moving poles considered in [ ]-[ ] in context of discrete symmetry transformation method. Let's remind that this transformation allows to directly construct new solutions from the known initial ones, i.e. if f is the solution of (11) then $F = D_s(f)$ is again solution of (11). Here D_s stands for discrete symmetry transformation. This transformation has a property

$$\det F_z = \det f_z$$

that is conserves a determinant . Comparing with (11c) makes encouraging to construct solutions of Ernst equations in that way and it will be a subject of further investigations.

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STASİONAR AKSİAL SİMMETRİK QRAVİTASİYA ƏSAS KİRAL SAHƏNİN MODELİ KİMİ

Aksional simmetrik qravitasiya tənlikləri (Ernst tənlikləri) hərəkət edən polyuslu əsas kiral sahənin tənliklərinə gətirilmişdir. Diskret simmetrik çevrilmə metodunun tətbiqi müzakirə edilmişdir.

М.А. Мухтаров

СТАЦИОНАРНАЯ АКСИАЛЬНО СИММЕТРИЧНАЯ ГРАВИТАЦИЯ КАК МОДЕЛЬ ГЛАВНОГО КИРАЛЬНОГО ПОЛЯ

Уравнения аксиально симметричной гравитации (уравнения Ернста) сведены к уравнениям главного кирального поля с подвижными полюсами. Обсуждается применение метода преобразований дискретных симметрий .

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