Klein Paradox in Modified Dirac and Salpeter Equations

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1. Introduction

It is well known, that Dirac equation in central symmetry field has unusual properties, like the leak of particle through the infinite wall, that means oscillation of solution in asymptote, for infinitly increasing potential, when the interaction is the zero component of Lorenz-vector. This unusual property is known as Klein Paradox [1]. The same lack has two particle Dirac equation [2,3].

By considering quarks and antiquarks the problem is being tried to solve by introducing additional scalar interaction [4,5]. So let us consider Dirac equation

$$(\vec{\alpha}\,\vec{p}+\beta m)\Psi(x)=(E-V)\Psi(x) \tag{I}$$

with infinitly increasing potential

$$\lim_{r\to\infty}V(r)=\infty$$

there wave function is represented as spinor

$$\Psi=\begin{pmatrix} a(r)\Omega_{jlm} \\ b(r)\Omega_{jl'm}\end{pmatrix}$$

we get radial equations

$$\begin{aligned}\frac{dA}{dr}+\frac{\chi}{r}A-(E-V+m)B=0\\ \frac{dB}{dr}-\frac{\chi}{r}B+(E-V-m)A=0\end{aligned} \tag{II}$$

there

$$A(r)=ra(r) \qquad B(r)=rb(r) \qquad \chi=l(l+1)-j(j+1)-\frac{1}{4}$$

After obvious transformations the equations (2) will look like

$$\frac{d^2A}{dr^2}-\frac{2\chi}{r^2}A+\left((E-V)^2-m^2\right)A=0$$

When $r\to\infty$, the second term can be neglected

$$\frac{d^2A(r)}{dr^2}+V^2A(r)=0 \tag{III}$$

The sign “+” shows, that there are no bound states. That means, the Klein Paradox takes place.

On the other hand, if we consider scalar potential in 4-space the Dirac Equation will look like

$$(\vec{\alpha}\,\vec{p}+\beta(m+S(r)))\Psi(\vec{r})=E\Psi(\vec{r}) \tag{IV}$$

After the transformation of corresponding radial equations we will get equation

$$\frac{d^2A(r)}{dr^2}-S^2(r)A(r)=0 \tag{V}$$

which has bound states and is free of Klein Paradox.

As a rule there are considered potentials containing both fields

$$U=aS(r)+bV(r) \qquad a,b=const \tag{VI}$$

In this case the existing of Klein Paradox depends on the correlation between two constants.

Besides that particular importance has equal mixture of scalar and vector potentials, which excluses spin-orbital interaction in Dirac equation [3,5] and Klein Paradox in two particle Dirac equation [2,3,6].

It would be natural to check the existing of this problem in field's quantum theory equations. I mean Bethe-Salpeter (BS) equation. There is an opinion that only vector interaction is not recommended here either because of Klein Paradox. But this fact is not proved yet. For example, in case of quark's confinement in hadrons the situation is not clear. There is not necessary to introduce an additional scalar field, it actually exists as potential's fourth component.

In last years a lot of works discused 3-dimensional relativistic equations which are results of BS equation's different reductions. 3-dimensional equations are interesting because they give us possibility to study quarkonium spectrum for different potentials. The great attention is paid to the formulation where by means of one particle mass rushing to infinity the problem reduces to Dirac equation of one particle in external field [7]. But in our opinion, not eveything is clear there. The information about second quantization could not be lost when we make one particle heavier. For instance, if we use 3-dimensional Kernel in BS equations the problem reduces to Salpeter equation which differs from two-particle Dirac equation in interaction term which contains projecting operator only on positive and only on negative frequencies. In this case this projecting operator is the only relict of second quantization.

In the 80-s was supposed, that equations of field's quantum theory must be free of Klein Paradox [3,6]. So Salpeter equation was studied from this point of view [8,9] by Krolikowski and Turski. But final conclusions were made based on Dirac modified equation, which is the result of making one particle infinitly heavy.

For example, the author of work [8] was studing the following Salpeter equation:

$$\left[E-\left(\vec{\alpha}^{(1)}\vec{p}+\beta^{(1)}m_1\right)-\left(-\vec{\alpha}^{(2)}\vec{p}+\beta^{(2)}m_2\right)-\Pi(\vec{p})V(r)\right]\Psi(\vec{r})=0 \qquad (1)$$

where $\Pi(\vec{p})$ is above mentioned projectiong operator:

$$\Pi(\vec{p})=\Lambda_+^{(1)}(\vec{p})\Lambda_+^{(2)}(-\vec{p})-\Lambda_-^{(1)}(\vec{p})\Lambda_-^{(2)}(-\vec{p}) \qquad (2)$$

and

$$\Lambda_\pm^{(i)}(\vec{p})=\frac{\omega_p\pm(\vec{\alpha}^{(i)}\vec{p}+\beta^{(i)}m_i)}{2\omega_p}\quad,\qquad \omega_p=\sqrt{\vec{p}^{\,2}+m^2}$$

are projecting operators on poitive and negative frequences for free spinors. They are nonlocal integral operators in coordinate space.

In the limit when $m_1=m_2=m\to\infty$ the kernel of this integral representation (Bessel function) becomes local $\delta(\vec{r}-\vec{r}\,')$ function and the equation (1) becomes local either. After this it is not difficult to determine asymptotic behaviour of wave function when $r\to\infty$. It will exponentialy fall for increasing potential V(r).

Turski considered the case of tending one particle's mass to infinity. Then equation (1) reduces to modified Dirac equation for light particle motion in "projected' field:

$$\left[\vec{\alpha}\,\vec{p}+\beta m+\Lambda_+(\vec{p})V(r)\right]\Psi(\vec{r})=E\Psi(\vec{r}) \qquad (3)$$

The author showed, by means of above mentioned integral representation and asymptotic restrictions on wave function and potential, that modified Dirac equation is free of Klein Paradox.

We consider, that all such statements are scanty because they are based on approximations and simplified assumptions about potential's behaviour. One is clear from these works, if modified Dirac equation is free of Klein Paradox the same is true for Salpeter equation.

In the following we consider our problem in momentum space and show that modified Dirac equation (3) has discrete spectrum if such spectrum has Schrödinger equation for the same potential V(r). This statement is equivalent to absence of Klein Paradox, by our opinion.

2. Dirac Modified Equation in Foldy-Wouthausen Representation

Consequently, we learn Dirac modified equation (3). If we act on this equation with projecting operator $\Lambda_{\pm}(\vec{p})$ in succession we get the following equations:

$$\left[E-(\vec{\alpha}\,\vec{p}+\beta m)\right]\Lambda_{+}(\vec{p})\Psi=\Lambda_{+}V\Psi \tag{4}$$

and

$$\Psi_{-}=\Lambda_{-}\Psi=0 \tag{5}$$

By using these equations (4) will look like:

$$\left[E-(\vec{\alpha}\,\vec{p}+\beta m)\right]\Lambda_{+}\Psi=\Lambda_{+}V\Lambda_{+}\Psi \tag{6}$$

Consequently, we have a problem of proper value with following hermitian Hamiltonian

$$H=\vec{\alpha}\,\vec{p}+\beta m+\Lambda_{+}V\Lambda_{+} \tag{7}$$

It is convenient to use Foldy-Wouthausen's transformation[11]:

$$e^{iS}\left(\vec{\alpha}\,\vec{p}+\beta m\right)e^{-iS}=\beta\omega_{p} \qquad\qquad \omega_{p}=\sqrt{\vec{p}^{2}+m^{2}} \tag{8}$$

It is clear, that

$$e^{iS}\Lambda_{+}(\vec{p})=\sqrt{\frac{2\omega_{p}}{\omega_{p}+m}}\frac{1}{2}(1+\beta)\Lambda_{+}(\vec{p})$$

$$\Lambda_{+}(\vec{p})e^{-iS}=\sqrt{\frac{2\omega_{p}}{\omega_{p}+m}}\Lambda_{+}(\vec{p})\frac{1}{2}(1+\beta) \tag{9}$$

For the transformed $\Psi_{FW}=e^{iS}\Psi$ function we get the following equation

$$(E-\beta\omega_{p})\Psi_{FW}=\sqrt{\frac{2\omega_{p}}{\omega_{p}+m}}\frac{1+\beta}{2}\Lambda_{+}(\vec{p})\int d^{3}kV\left(\vec{p}-\vec{k}\right)\Lambda_{+}\left(\vec{k}\right)\frac{1+\beta}{2}\sqrt{\frac{2\omega_{k}}{\omega_{k}+m}}\Psi_{FW}\left(\vec{k}\right) \tag{10}$$

It is natural to represent Ψ_{FW} as a two component spinor

$$\Psi_{FW}=\begin{pmatrix}\varphi\\ \chi\end{pmatrix} \tag{11}$$

Then equation (10) will give us the following system:

$$(E-\beta\omega_{p})\varphi(\vec{p})=\sqrt{\frac{2\omega_{p}}{\omega_{p}+m}}\frac{1+\beta}{2}\Lambda_{+}(\vec{p})\int d^{3}kV\left(\vec{p}-\vec{k}\right)\Lambda_{+}\left(\vec{k}\right)\sqrt{\frac{2\omega_{k}}{\omega_{k}+m}}\varphi\left(\vec{k}\right)$$

$$(E+\omega_{p})\chi(\vec{p})=0 \tag{12}$$

The second equation does not have untrivial solutions, so far as $E\neq-\omega_{p}$, $\chi=0$

Now we can calculate the matrix structure of the right side of the equation (12) in evident form. After simple transformations we get

$$\left(E-\omega_{p}\right)\varphi(\vec{p})=\sqrt{\frac{2\omega_{p}}{\omega_{p}+m}}\int d^{3}k\left[\frac{\omega_{p}+m}{2\omega_{p}}V\left(\vec{p}-\vec{k}\right)\frac{\omega_{k}+m}{2\omega_{k}}+\frac{\vec{\sigma}\,\vec{p}}{2\omega_{p}}V\left(\vec{p}-\vec{k}\right)\frac{\vec{\sigma}\,\vec{k}}{2\omega_{k}}\right]\sqrt{\frac{2\omega_{k}}{\omega_{k}+m}}\varphi\left(\vec{k}\right) \tag{13}$$

3. The Radial Form of Modified Dirac Equation

Let us analyse the equation (13) by angles. For this purpose we can use basis of spherical spinors [12]:

$$\varphi(\vec{p})=f(p)\Omega_{jlm}(\vec{n}_{p}) \qquad\qquad \vec{n}_{p}=\frac{\vec{p}}{|\vec{p}|} \tag{14}$$

where $\Omega_{jl\,m}$ functions satisfy following equation

$$\left(\vec{\sigma}\,\vec{n}_{p}\right)\Omega_{jlm}(\vec{n}_{p})=-\Omega_{jl'm}(\vec{n}_{p}) \qquad\qquad (l+l'=2j) \tag{15}$$

In explicit form this functions are expressed by spherical harmonics and corresponding Klebsh-Gordon coefficients:

$$\Omega_{jlM}(\vec{n}_p)=\begin{pmatrix} C^{jM}_{l,M-1/2,1/2,1/2}\, Y_{l,M-1/2}(\vec{n}) \\ C^{jM}_{l,M+1/2,1/2,-1/2}\, Y_{l,M+1/2}(\vec{n}) \end{pmatrix} \tag{16}$$

Therefore, if we express $V(\vec{p}-\vec{k})$ potential as series of spherical harmonics

$$V(\vec{p}-\vec{k})=\sum_{l=0}^{\infty}\sum_{m'=-l}^{l} v_l(p,k) Y_{lm'}(\vec{n}_p) Y^{*}_{lm'}(\vec{n}_k) \tag{17}$$

and insert everything mentioned above in the equation (13) and use properties of orthogonality the equation will not depend on angles. So we get the following radial equation

$$(E-\omega_p)f(p)=\int_0^{\infty} k^2 dk V_l(p,k)\chi(p,k)f(k) \tag{18}$$

where $\chi(p,k)$ is an additional factor, which is the result of projecting operator's existing:

$$\chi(p,k)=\sqrt{\frac{\omega_p+m}{2\omega_p}}\left[1+\frac{pk}{(\omega_p+m)(\omega_k+m)}\right]\sqrt{\frac{\omega_k+m}{2\omega_k}} \tag{19}$$

Equation (18) is our main result.

4. Properties of Radial Equation Spectrum

The kind of spectrum of radial (integral) equation (18) depends on the properties of it's kernel, which now contains FW-factor $\chi(p,k)$ together with radial component of potential. Just this factor expresses nonlocality of effective potential.

First of all let us discuss some properties of $\chi(p,k)$ factor:

1) It loks like the sum of two members factorized by p and k variables. This can be important by studying various factorized potentials.
2) When k=p (on the energetical surface in scattering problem) $\chi=1$.
3) It is positive-definite and bounded when $p,k\to\infty$.
4) It has not any singularity for physical values of variables.
5) To reach the nonrelativistic limit in equation (18) it is enough to expand it into a series of p^2 and k^2. This operation does not affect the smallness of $v_l(p,k)$ potential.

Based on the properties mentioned above we can answer our main question – does Klein Paradox take place or not?

It is clear, that integral equation (18) without $\chi(p,k)$ is Salpeter spinless equation's radial form in momentum space. As it is known [10,13] Salpeter spinless equation for unlimited, increased potential has confinement type solutions – only discrete spectrum. Because of above mentioned properties of $\chi(p,k)$ the kernel of equation (18) is Fredholm-Smith type. It is multiplied on limited nonsingular function, according to well known theorem [14,15] the kernel remains Fredholm-Smith type and therefore the spectrum of equation(18) will be only discrete if Schrödinger (or spinless Salpeter) equation has discrete spectrum for the same potential.

In our opinion, this conclusion is equivalent to absence of Klein Paradox in modified Dirac (and Salpeter) equation.

It is strange that in our equation (13) remains dependent on orbital momentum only. That means, projecting operator Λ_+ separates spin-orbit coupling.

By introducing additional scalar interaction the same result is obtained for equal mixture of scalar an vector potentials.

Consequently, all above mentioned results can be naturally obtained be considering vector potential only. That is why it is not necessary to introduce the scalar potential artificially.

5. References

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