HAMILTONIAN-TYPE OPERATORS AND q-ORTHOGONAL POLYNOMIALS

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Abstract

It is known that self-adjoint operators, which belong to a certain class of operators for the discrete series representations of the quantum algebra $\mathrm{su}_q(1,1)$, may serve as Hamiltonians of some physical systems. These operators are expressed in the canonical basis by Jacobi matrices. The problem of diagonalization of these operators (eigenfunctions, spectra, overlap coefficients, etc.) is solved for a wide class of such operators by using their connection with the theory of q-orthogonal polynomials.

1. Introduction

It is well known that all eigenvalues of self-adjoint operators are real. This mathematical fact has been vital for formulating one of the fundamental constructive postulates of quantum mechanics: the variables used for describing a dynamical system in quantum mechanics are represented by self-adjoint operators; the values, which a given variable can take, correspond to observable physical quantities and they are found as eigenvalues of the associated self-adjoint operators (see, for example, [1]). So, the study of explicit instances of relevant self-adjoint operators, related to particular physical systems, is of clear interest. To be more specific, we refer to recent studies in quantum optics (see [2] and references therein), where it was shown that many models, such as Raman and Brillouin scattering, parametric conversion and the interaction of two-level atoms with a single mode radiation field (Dicke model), can be described by interaction Hamiltonians, which are certain self-adjoint difference operators from representations of the quantum algebra $\mathrm{su}_q(2)$.

In this paper we wish to discuss yet another self-adjoint difference operators, which may serve as Hamiltonians of some physical systems. They belong to a certain class of operators for the discrete series representations of the quantum algebra $\mathrm{su}_q(1,1)$ and can be expressed in the canonical basis by a Jacobi matrix. We show that the problem of diagonalization of such operators (eigenfunctions, spectra, overlap coefficients, etc.) is solved by using the connection of these operators with the theory of q-orthogonal polynomials. The point is that an orthogonality measure for a family of q-orthogonal polynomials, associated with a certain Hamiltonian operator, determines a spectrum of this operator and its spectral decomposition.

It is essential to note that our study of Hamiltonian-type operators was first started in [3] and [4], and then has been developed in [5–7]. In particular, in [7] we considered examples of Hamiltonians for physical systems in non-commutative world (see in this connection [8]).

A summary of what the remaining sections contains is as follows. Section 2 gives some useful results on the quantum algebra $\mathrm{su}_q(1,1)$ and its discrete series representations. In section 3 we detail general properties of those Hamiltonian-type operators, which are operators of the discrete series of the algebra $\mathrm{su}_q(1,1)$. In sections 4 and 5, some examples of Hamiltonians with bounded continuous spectra are studied. Section 6 is devoted to Hamiltonians with bounded discrete spectra.

Throughout the sequel we always assume that q is a fixed positive number such that $q < 1$. We extensively use the theory of q-special functions and notations of the standard q-analysis (see, for example, [9]). In particular, we assume that

$$[a]_q := \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}}\,, \tag{1.1}$$

where a can be a number or an operator.

2. The algebra $\mathrm{su}_q(1,1)$ and its discrete series representations

The classical Lie algebra $\mathrm{su}(1,1)$ is generated by the elements $J_0^{\rm cl}, J_1^{\rm cl}, J_2^{\rm cl}$, satisfying the relations

$$[J_0^{\rm cl}, J_1^{\rm cl}] = {\rm i}J_2^{\rm cl}\,, \qquad [J_1^{\rm cl}, J_2^{\rm cl}] = -{\rm i}J_0^{\rm cl}\,, \qquad [J_2^{\rm cl}, J_0^{\rm cl}] = {\rm i}J_1^{\rm cl}\,.$$

In terms of the raising and lowering operators $J_\pm^{\rm cl} = J_1^{\rm cl} \pm {\rm i}J_2^{\rm cl}$ these commutation relations can be written as

$$[J_0^{\rm cl}, J_\pm^{\rm cl}] = \pm J_\pm^{\rm cl}\,, \qquad [J_-^{\rm cl}, J_+^{\rm cl}] = 2J_0^{\rm cl}\,.$$

The discrete series representations T_l^+ of $\mathrm{su}(1,1)$ with lowest weights are given by a positive number l and they are realized on the spaces $\mathcal{L}_l$ of polynomials in x. The basis in $\mathcal{L}_l$ consists of the monomials

$$g_n^l(x) = \{(2l)_n/n!\}^{1/2} x^n\,, \qquad n = 0, 1, 2, 3, \cdots\,.$$

Assuming that this basis consists of orthonormal elements, one defines a scalar product in $\mathcal{L}_l$. The closure of $\mathcal{L}_l$ leads to a Hilbert space, on which the representation T_l^+ acts.

We consider an explicit realization of the representation operators $J_i^{\rm cl}$, $i = 0, 1, 2$, in terms of the first-order differential operators:

$$J_0^{\rm cl} = x\frac{d}{dx} + l\,, \qquad J_1^{\rm cl} = \frac{1}{2}(1+x^2)\frac{d}{dx} + lx\,, \qquad J_2^{\rm cl} = \frac{i}{2}(1-x^2)\frac{d}{dx} - ilx\,. \tag{2.1}$$

Then

$$J_0^{\rm cl}\, g_n^l = (l+n)g_n^l, \quad J_+^{\rm cl}\, g_n^l = \sqrt{(2l+n)(n+1)}\,g_{n+1}^l, \quad J_-^{\rm cl}\, g_n^l = \sqrt{(2l+n-1)n}\,g_{n-1}^l.$$

In some physical models a Hamiltonian is represented by the self-adjoint operator $J_0^{\rm cl} - J_1^{\rm cl}$. For instance, the n-dimensional harmonic oscillator in nonrelativistic quantum mechanics is governed by the Hamiltonian

$$H(x) = \frac{\hbar\omega}{2}\sum_{k=1}^{n}\left(\xi_k^2 - \frac{d^2}{d\xi_k^2}\right) = \hbar\omega\left[\sum_{k=1}^{n} a^+(x_k)\,a(x_k) + \frac{n}{2}\right],$$

where $x = \{x_1, x_2, \cdots, x_n\}$, the dimensionless variables $\xi_k = \sqrt{m\omega/\hbar}\,x_k$, $k = 1, 2, \cdots, n$, and the annihilation and creation boson operators $a(x_k) = \frac{1}{\sqrt{2}}(\xi_k + d/d\xi_k)$ and $a^+(x_k) = \frac{1}{\sqrt{2}}(\xi_k - d/d\xi_k)$, respectively, satisfy the Heisenberg commutation relations

$$[a(x_k), a^+(x_j)] = \delta_{kj}, \quad k, j = 1, 2, \cdots, n.$$

The dynamical symmetry algebra for this model is known to be the Lie algebra $\mathrm{su}(1,1)$ with the generators

$$J_0(x) = \frac{1}{2}\sum_{k=1}^{n} a^+(x_k)\,a(x_k), \quad J_1(x) = \frac{1}{2}\sum_{k=1}^{n} a^2(x_k), \quad J_2(x) = \frac{1}{2}\sum_{k=1}^{n}[a^+(x_k)]^2. \tag{2.2}$$

One readily verifies that, for example, the n-dimensional Laplace operator $\Delta = \sum_{k=1}^{n} d^2/d\xi_k^2$ in terms of the su$(1,1)$-generators (2.2) is just $2[J_1(x) - J_0(x)] - n/2$. Hence, the problem of funding spectrum and eigenfunctions of the Laplace operator Δ in this n-dimensional Euclidean space is equivalent to that of the operator $J_1(x) - J_0(x)$.

Now we return to the realization (2.1). It is not hard to show that the eigenfunctions of the operator $J_0^{\rm cl} - J_1^{\rm cl}$ are of the form

$$[J_0^{\rm cl} - J_1^{\rm cl}]\, \eta_\lambda^l(x) = \lambda\, \eta_\lambda^l(x)\,, \tag{2.3}$$

$$\eta_\lambda^l(x) = (1-x)^{-2l} \exp\left(\frac{2\lambda x}{x-1}\right)\,. \tag{2.4}$$

The quantum algebra su$_q(1,1)$ and its irreducible representations are obtained by deformation of the corresponding relations for the Lie algebra su$(1,1)$ and its irreducible representations. The algebra su$_q(1,1)$ is defined as the associative algebra, generated by the elements J_+, J_-, and J_0, which satisfy the commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_-, J_+] = \frac{q^{J_0} - q^{-J_0}}{q^{1/2} - q^{-1/2}} \equiv [2J_0]_q\,, \tag{2.5}$$

and the conjugation relations $J_0^* = J_0$, $J_+^* = J_-$.

We are interested in the discrete series representations of su$_q(1,1)$ with lowest weights. These irreducible representations will be denoted by T_l^+, where l is a lowest weight, which can be any positive number (see, for example, [10]). These representations are obtained by deforming the corresponding representations of the Lie algebra su$(1,1)$.

As in the classical case, the representation T_l^+ can be realized on the space $\mathcal{L}_l$ of all polynomials in x. We choose a basis for this space, consisting of the monomials

$$f_n^l \equiv f_n^l(x) := c_n^l\, x^n, \quad n = 0, 1, 2, \cdots, \tag{2.6}$$

where

$$c_0^l = 1, \qquad c_n^l = \prod_{k=1}^{n} \frac{[2l+k-1]_q^{1/2}}{[k]_q^{1/2}} = q^{(1-2l)n/4}\, \frac{(q^{2l};q)_n^{1/2}}{(q;q)_n^{1/2}}\,, \quad n = 1, 2, 3, \cdots, \tag{2.7}$$

and $(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$. The representation T_l^+ is then realized by the operators

$$J_0 = x\frac{d}{dx} + l, \qquad J_\pm = x^{\pm 1}[J_0 \pm l]_q\,. \tag{2.8}$$

As a result of this realization, we have

$$J_0\, f_n^l = (l+n)\, f_n^l, \quad J_+\, f_n^l = \sqrt{[2l+n]_q\, [n+1]_q}\, f_{n+1}^l, \tag{2.9}$$

$$J_-\, f_n^l = \sqrt{[2l+n-1]_q\, [n]_q}\, f_{n-1}^l. \tag{2.10}$$

We know that the discrete series representations T_l^+ can be realized in a Hilbert space, on which the conjugation relations $J_0^* = J_0$ and $J_+^* = J_-$ are satisfied. In order to obtain such a Hilbert space, we assume that the monomials $f_n^l(x)$, $n = 0, 1, 2, \cdots$, constitute an orthonormal basis for this Hilbert space. This introduces a scalar product $\langle \cdot, \cdot \rangle$ into the space $\mathcal{L}_l$. Then we close this space with respect to this scalar product and obtain a Hilbert space, which will be denoted by $\mathcal{H}_l$. The Hilbert space $\mathcal{H}_l$ consists of functions (series)

$$f(x) = \sum_{n=0}^{\infty} b_n f_n^l(x) = \sum_{n=0}^{\infty} b_n c_n^l x^n = \sum_{n=0}^{\infty} a_n x^n,$$

where $a_n = b_n c_n^l$. Since $\langle f_m^l, f_n^l \rangle = \delta_{mn}$ by definition, for $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $\tilde{f}(x) = \sum_{n=0}^{\infty} \tilde{a}_n\, x^n$ we have $\langle f, \tilde{f} \rangle = \sum_{n=0}^{\infty} a_n\, \tilde{a}_n / |c_n^l|^2$, that is, the Hilbert space $\mathcal{H}_l$ consists of analytical functions $f(x) = \sum_{n=0}^{\infty} a_n\, x^n$, such that

$$\|f\|^2 \equiv \sum_{n=0}^{\infty} |a_n / c_n^l|^2 < \infty.$$

It is directly checked that for a function $f(x) \in \mathcal{H}_l$ we have $q^{cx\frac{d}{dx}}\, f(x) = f(q^c x)$. Therefore, taking into account formulas (2.8), we conclude that

$$q^{J_0/2}\, f(x) = q^{\frac{1}{2}(x\frac{d}{dx}+l)}\, f(x) = q^{l/2}\, f(q^{1/2}x)\,, \tag{2.11}$$

$$J_+\, f(x) = \frac{x}{q^{1/2} - q^{-1/2}} \left[q^l f(q^{1/2}x) - q^{-l} f(q^{-1/2}x) \right]\,, \tag{2.12}$$

$$J_-\, f(x) = \frac{1}{(q^{1/2} - q^{-1/2})x} \left[f(q^{1/2}x) - f(q^{-1/2}x) \right]\,. \tag{2.13}$$

3. Hamiltonian operators and q-orthogonal polynomials

We are interested in spectra, eigenfunctions and overlap functions for operators in the representations T_l^+, which correspond to elements of the quantum algebra $\mathrm{su}_q(1,1)$ of the form

$$H := q^{p\, J_0} (J_+ + J_-)\, q^{p\, J_0} + f(q^{J_0}), \quad p \in \mathbb{R}, \tag{3.1}$$

where f is some polynomial (or function). These operators have the following properties:

(i) They are representable in the basis (2.6) by a Jacobi matrix.
(ii) They are symmetric operators.
(iii) They are not necessarily self-adjoint operators.

Recall that a Jacobi matrix is a matrix, with all entries vanishing except for those which occur on the main diagonal and on two neighboring (upper and lower) diagonals.

Note that in the case (iii) a symmetric operator has self-adjoint extensions. These extensions can serve as Hamiltonian operators. However, finding of self-adjoint extensions is a complicated problem in the each particular case.

The most important example of the operators of the form (3.1) is

$$H^{(p)} := q^{pJ_0/4} (J_+ + J_-) q^{pJ_0/4}, \quad p \in \mathbb{R}.$$

For these operators the following theorem is true [11].

Theorem 1. *If $p > 1$, then the operator $H^{(p)}$ is bounded and has a discrete simple spectrum. Zero is a unique point of accumulation of the spectrum. If $p < 1$, then the closure $\overline{H^{(p)}}$ of the symmetric operator $H^{(p)}$ is not a self-adjoint operator and it has deficiency indices (1,1), that is, $\overline{H^{(p)}}$ has a one-parameter family of self-adjoint extensions. These extensions* *have discrete simple spectra. If $p = 1$, then $H^{(p)}$ has a continuous simple spectrum, which covers the interval $(-b, b)$, $b = 2/(q^{-1/2} - q^{1/2})$.*

This theorem describes spectra of the operators $\overline{H^{(p)}}$ and does not explain how to find eigenfunctions. To find them, one can employ the theory of q-orthogonal polynomials.

There exist close relationship between the following directions, which we exploit to study Hamiltonian-type operators:

(i) the theory of symmetric operators L, representable by a Jacobi matrix;
(ii) the theory of orthogonal polynomials;
(iii) the theory of classical moment problem

(see [12] and [13]). Let us describe briefly this relationship. Let L be a closed symmetric operator on a Hilbert space $\mathcal{H}$. Let $e_1, e_2, \cdots$ be a basis in $\mathcal{H}$ such that

$$L\, e_n = a_n\, e_{n+1} + b_n\, e_n + a_{n-1}\, e_{n-1}.$$

Let $|x\rangle = \sum_{n=0}^{\infty} p_n(x) e_n$ be an eigenvector of L with the eigenvalue x, that is, $L|x\rangle = x|x\rangle$. Then

$$L\,|x\rangle = \sum_{n=0}^{\infty} [\, p_n(x) a_n e_{n+1} + p_n(x) b_n e_n + p_n(x) a_{n-1} e_{n-1}] = x \sum_{n=0}^{\infty} p_n(x) e_n.$$

Equating coefficients of the vector e_n, one comes to a recurrence relation for the coefficients $p_n(x)$:

$$a_n\, p_{n+1}(x) + b_n\, p_n(x) + a_{n-1}\, p_{n-1}(x) = x\, p_n(x).$$

Since $p_{-1}(x) = 0$, then setting $p_0(x) \equiv 1$, we see that this relation completely determines the coefficients $p_n(x)$. Moreover, $p_n(x)$ are polynomials in x of degree n. If coefficients a_n and b_n are real, then all coefficients of the polynomials $p_n(x)$ are real and they are orthogonal with respect to some positive measure $\mu(x)$. If the operator L is self-adjoint, then this measure is uniquely determined and

$$\int p_m(x) p_n(x) d\mu(x) = \delta_{mn},$$

where the integration is taken over some subset (possibly discrete) of $\mathbb{R}$. Moreover, the spectrum of the operator L is simple and coincide with the set, on which the polynomials are orthogonal. The measure $\mu(x)$ determines the spectral decomposition of the operator L (for details see [13], Chapter VII).

If a closed symmetric operator L is not self-adjoint, then the measure $\mu(x)$ is not determined uniquely. Moreover, in this case there exist infinitely many measures, with respect to which the polynomials are orthogonal. Among these measures there are so-called extremal measures (that is, such that a set of polynomials $\{p_n(x)\}$ is complete in the Hilbert space L^2 with respect to the corresponding measure). These measures determine self-adjoint extensions of the symmetric operator L.

On the other hand, with the polynomials $p_n(x)$, $n = 0, 1, 2, \cdots$, the classical moment problem is associated [12]. Namely, with these polynomials (that is, with the coefficients a_n and b_n in the corresponding recurrence relation) positive numbers c_n, $n = 0, 1, 2, \cdots$, are related, which enter into the classical moment problem. The definition of classical moment problem consists in the following. One looks for a measure $\mu(x)$, such that

$$\int x^n d\mu(x) = c_n, \qquad n = 0, 1, 2, \cdots, \tag{4.1}$$

where the c_n, $n = 0, 1, 2, \cdots$ are some positive numbers and integration is taken over some fixed subset of $\mathbb{R}$. There are then two main questions:

(i) Does exist a measure $\mu(x)$, such that relations (4.1) are satisfied?
(ii) If such a measure exists, is it determined uniquely?

If the answer to the first question is positive, then the numbers c_n, $n = 0, 1, 2, \cdots$, are those, which correspond to a particular family of orthogonal polynomials. Moreover, a measure $\mu(x)$ then coincides with the measure, with respect to which these polynomials are orthogonal.

If a measure in (4.1) is determined uniquely, then we say that we deal with determinate moment problem. It is the case when the region of integration is bounded. If there are many measures, with respect to which relations (4.1) hold, then we say that we deal with indeterminate moment problem. In this case there exist infinitely many measures $\mu(x)$, for which (4.1) takes place. In the second case the corresponding polynomials are orthogonal with respect to all these measures and the corresponding symmetric operator L is not self-adjoint.

Thus, we see that one can study the operator L by investigating the corresponding sets of orthogonal polynomials and their moment problems. To illustrate that, we confine our attention below to certain examples of Hamiltonian-type operators.

4. Hamiltonians with bounded continuous spectra

In this section we are interested in the operator

$$H_1 := \frac{a}{2}\, q^{J_0} - b - \frac{1}{2}\,(q^{1/4} J_+ + q^{-1/4} J_-)\, q^{J_0/2} \tag{5.1}$$

of the discrete series representation T_l^+, where

$$a = (q^{1/4} + q^{-1/4})\, b, \quad b = (q^{1/2} - q^{-1/2})^{-1}\,.$$

The representation operator $(q^{1/4}J_+ + q^{-1/4}J_-)q^{J_0/2}$ is bounded. Since J_0 has the eigenvalues $m = l, l+1, l+2, \cdots,$ the operator q^{J_0} is also bounded (recall that $0 < q < 1$). Thus, the operator H_1 is bounded. It is easy to check that H_1 is a self-adjoint operator since

$$H_1\, f_k^l = \beta_k\, f_k^l - \alpha_k\, f_{k+1}^l - \alpha_{k-1}\, f_{k-1}^l\,,$$

where

$$\alpha_k := \frac{1}{2}\left\{ q^{l+k+1/2}\,[2l+k]_q\,[k+1]_q \right\}^{1/2}, \quad \beta_k = \frac{(q^{1/4}+q^{-1/4})q^{l+k} - 2}{2(q^{1/2}-q^{-1/2})}.$$

The constants a and b in (5.1) are chosen in such a way that in the limit as $q \to 1$ the operator H_1 reduces to the $\mathrm{su}_{1,1}$-operator $J_0^{\rm cl} - J_1^{\rm cl}$ (see formula (2.3)).

Eigenfunctions of the operator H_1,

$$H_1\, \xi_\lambda^l(x) = \lambda(q)\, \xi_\lambda^l(x)\,, \tag{5.2}$$

and its spectrum can be found by using the explicit realization (2.8) for the generators J_0 and $J_\pm$. Indeed, from formulas (2.11)–(2.13) it follows that

$$H_1\, f(x) = \frac{b}{2x}\, q^{(2l-1)/4}\,[(1-q^{(1-2l)/4}\,x)^2\, f(x) - (1-q^{l/2}\,x)(1-q^{(l+1)/2}\,x)\, f(qx)] \tag{5.3}$$

for an arbitrary function $f(x)$. It is thus natural to look for the eigenfunctions $\xi_\lambda^l(x)$ of the operator H_1 in the form

$$\xi_\lambda^l(x) = \frac{(\alpha x;q)_\infty\,(\beta x;q)_\infty}{(\gamma x;q)_\infty(\delta x;q)_\infty}\,, \tag{5.4}$$

where $(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r)$. Since $(a;q)_\infty = (1-a)(aq;q)_\infty$ by definition, we have

$$\xi_\lambda^l(qx) = \frac{(1-\gamma x)(1-\delta x)}{(1-\alpha x)(1-\beta x)}\,\xi_\lambda^l(x;q)\,. \tag{5.5}$$

Substituting (5.4) and (5.5) into (5.3), one gets

$$H_1\xi_\lambda^l(x) = \frac{b}{2x}q^{(2l-1)/4}\left\{(1-q^{(1-2l)/4}x)^2 - (1-q^{l/2}x)(1-q^{(l+1)/2}x)\frac{(1-\gamma x)(1-\delta x)}{(1-\alpha x)(1-\beta x)}\right\}\xi_\lambda^l(x).$$

This equation can be written as

$$H_1\,\xi_\lambda^l(x) = \frac{b}{2}\,q^{(2l-1)/4}\,\frac{Ax^3+Bx^2+Cx+D}{(1-\alpha x)(1-\beta x)}\,\xi_\lambda^l(x), \tag{5.6}$$

where the constant coefficients A, B, C, and D are equal to

$$A = q^{1/2}(q^{-l}\alpha\beta - q^l\gamma\delta),$$

$$B = q^{1/2}[q^l(\gamma+\delta) - q^{-l}(\alpha+\beta)] + (1+q^{1/2})q^{l/2}\gamma\delta - 2q^{(1-2l)/4}\alpha\beta,$$

$$C = \alpha\beta - \gamma\delta + 2q^{(1-2l)/4}(\alpha+\beta) - (1+q^{1/2})q^{l/2}(\gamma+\delta) - q^{1/2}\,(q^l - q^{-l}),$$

$$D = \gamma+\delta-\alpha-\beta + (1+q^{1/2})q^{l/2} - 2q^{(1-2l)/4}.$$

It is clear from (5.6) that the $\xi_\lambda^l(x)$ is an eigenfunction of the operator H_1 if the factor in front of $\xi_\lambda^l(x)$ on the right-hand side of (5.6) does not depend on x. It is the case if

$$A = 0, \qquad B = \alpha\beta D, \qquad C = -(\alpha+\beta)D\,. \tag{5.7}$$

Then eigenvalues of the operator H_1 on the right-hand side of (5.6) will be equal to $\lambda = q^{(2l-1)/4}D/2(q^{1/2} - q^{-1/2})$. Requirements (5.7) are equivalent to the following three relations between the parameters $\alpha, \beta, \gamma, \delta$:

$$\alpha\beta = q^{2l}\gamma\delta, \quad (q^{1/2-l} - \alpha\beta)(\alpha+\beta) = (q^{l+1/2} - \alpha\beta)(\gamma+\delta) - (1+q^{1/2})q^{-l/2}(q^l - q^{-l})\alpha\beta,$$

$$(q^l - q^{-l})(q^{1/2} - q^{-l}\alpha\beta) = [\alpha+\beta-(1+q^{1/2})q^{l/2}](\gamma+\delta-\alpha-\beta).$$

From these relations it follows that

$$\alpha = q^{l/2}, \quad \beta = q^{(l+1)/2}, \quad \gamma = q^{(1-2l)/4}\,e^{\mathrm{i}\theta}, \quad \delta = q^{(1-2l)/4}\,e^{-\mathrm{i}\theta},$$

where θ is an arbitrary angle. Consequently, the eigenfunctions of the operator H_1 are equal to

$$\xi_\lambda^l(x) = \frac{(q^{l/2}x;q)_\infty\,(q^{(l+1)/2}x;q)_\infty}{(q^{(1-2l)/4}e^{\mathrm{i}\theta}x;q)_\infty\,(q^{(1-2l)/4}e^{-\mathrm{i}\theta}x;q)_\infty}, \tag{5.8}$$

and the corresponding eigenvalues are

$$\lambda \equiv \lambda(q) = (1-\cos\theta)/(q^{-1/2} - q^{1/2})\,.$$

The eigenfunctions (5.8) are in fact the generating functions for the continuous q-Laguerre polynomials

$$P_n^{(\alpha)}(y|q) = (q^{(2\alpha+3)/4}e^{-\mathrm{i}\theta};q)_n(q;q)_n^{-1}\,q^{(2\alpha+1)n/4}e^{\mathrm{i}n\theta}$$

$$\times{}_2\phi_1(q^{-n},\ q^{(2\alpha+1)/4}e^{\mathrm{i}\theta};\ q^{-n-(2\alpha-1)/4}e^{\mathrm{i}\theta}\ ;\ q; q^{-(2\alpha-1)n/4}e^{-\mathrm{i}\theta}),$$

where $y = \cos\theta$ and ${}_2\phi_1$ is the basic hypergeometric function, defined by formula (1.2.14) in [9]. In order to make this evident, one needs to represent (5.8) in the form

$$\xi^l_\lambda(x;q) = \frac{(q^{2l-1/2}ax;q)_\infty (q^{2l}ax;q)_\infty}{(q^{l-1/4}e^{\mathrm{i}\theta}ax;q)_\infty (q^{l-1/4}e^{-\mathrm{i}\theta}ax;q)_\infty}, \tag{5.9}$$

where $a = q^{(1-3l)/2}$. Consequently, due to formula (3.19.11) in [14], the desired connection is

$$\xi^l_\lambda(x) = \sum_{n=0}^{\infty} q^{n(1-3l)/2}\, P_n^{(2l-1)}(\cos\theta|q)\, x^n = \sum_{n=0}^{\infty} \frac{q^{n(1-3l)/2}}{c^l_n}\, P_n^{(2l-1)}(\cos\theta|q)\, f^l_n(x),$$

where $\cos\theta = 1 - (q^{-1/2} - q^{1/2})\lambda$.

Thus, we have proved that the eigenfunctions $\xi^l_\lambda(x)$ are connected with the basis elements $f^l_n(x)$ by the formula

$$\xi^l_\lambda(x) = \sum_{n=0}^{\infty} p_n(\lambda) f^l_n(x), \tag{5.10}$$

where the overlap coefficients $p_n(\lambda)$ are explicitly given by

$$p_n(\lambda) = q^{(1/4-l)n} (q;q)_n^{1/2} (q^{2l};q)_n^{-1/2} P_n^{(2l-1)}(1-(q^{-1/2}-q^{1/2})\lambda|q). \tag{5.11}$$

To find a spectrum of H_1, we take into account the following. The self-adjoint operator H_1 is represented by a Jacobi matrix in the basis $f^l_n(x)$, $n = 0, 1, 2, \cdots$, with nonvanishing entries α_k and β_k. As is evident from (5.10), the eigenfunctions $\xi^l_\lambda(x)$ are expanded in the basis elements $f^l_n(x)$ with the coefficients (5.11). According to the results of Chapter VII in [13], these polynomials $p_n(\lambda)$ are orthogonal with respect to some measure $d\mu(\lambda)$ (this measure is unique, up to a multiplicative constant, since the operator H_1 is bounded). The set (a subset of $\mathbb{R}$), on which the polynomials are orthogonal, coincides with the spectrum of the operator H_1 and $d\mu(\lambda)$ determines the spectral measure of this operator; the spectrum of H_1 is simple (see Chapter VII in [13]).

We remind the reader that the orthogonality relation for the continuous q-Laguerre polynomials $P_n^{(2l-1)}(y|q)$ has the form

$$\frac{1}{2\pi}\int_{-1}^{1} P_m^{(2l-1)}(y|q) P_n^{(2l-1)}(y|q) \frac{w(y)dy}{\sqrt{1-y^2}} = \frac{(q^{2l};q)_n\, q^{(2l-1/2)n}}{(q;q)_\infty (q^{2l};q)_\infty (q;q)_n}\, \delta_{mn},$$

where

$$w(y) = \left| \frac{(e^{\mathrm{i}\theta};q^{1/2})_\infty (-e^{\mathrm{i}\theta};q^{1/2})_\infty}{(q^{l-1/4}e^{\mathrm{i}\theta};q^{1/2})_\infty} \right|^2, \quad y = \cos\theta$$

(see formula (3.19.2) in [14]). Therefore, the orthogonality relation for the overlap coefficients (5.11) is

$$\int_0^{2q^{1/2}/(1-q)} p_m(\lambda) p_n(\lambda) \hat{w}(\lambda) d\lambda = \delta_{mn}, \tag{5.12}$$

where

$$\hat{w}(\lambda) = \frac{1}{2\pi}(q;q)_\infty (q^{2l};q)_\infty \sqrt{\frac{1-q}{\lambda q^{1/2}}}\, \frac{w(1-q^{-1/2}(1-q)\lambda)}{\sqrt{2-q^{-1/2}(1-q)\lambda}}. \tag{5.13}$$

Consequently, the spectrum of the operator H_1 (that is, a range of λ) coincides with the finite interval $[0,\, 2q^{1/2}/(1-q)]$. The spectrum is continuous and simple. The continuity of the spectrum means that the eigenfunctions $\xi^l_\lambda(x)$ do not belong to the Hilbert space $\mathcal{H}_l$. They belong to the space of functionals on $\mathcal{L}_l$, which can be considered as a space of generalized functions on $\mathcal{L}_l$. We have thus proved the following theorem.

Theorem 2. *The self-adjoint operator H_1 has the continuous and simple spectrum, which covers the finite interval $[0,\, 2q^{1/2}/(1-q)]$. The eigenfunctions $\xi^l_\lambda(x)$ are explicitly given by (5.8) and they are related to the basis (2.6) by formula (5.10).*

As we remarked at the beginning of this section, the operator H_1 represents a q-extension of the $\mathrm{su}_{1,1}$-operator $J_0^{\rm cl} - J_1^{\rm cl}$. In the limit as $q \to 1$ the finite interval $[0,\, 2q^{1/2}/(1-q)]$ of the eigenvalues of the operator H_1 extends to the infinite interval $[0, \infty)$. So if one puts $\cos\theta = q^\mu$, then $\lim_{q\to 1} \lambda(q^\mu) = \mu$.

Besides, it is known that the continuous q-Laguerre polynomials $P_n^{(\alpha)}(y|q)$ have the following limit property $\lim_{q\to 1} P_n^{(\alpha)}(q^\lambda|q) = L_n^{(\alpha)}(2\lambda)$ (see [14], formula (5.19.1)). Thus, the coefficients of the series expansion (5.10) in x of the eigenfunctions $\xi^l_\lambda(x)$, $\cos\theta = q^\mu$, coincide with the coefficients of the corresponding expansion of the $\mathrm{su}_{1,1}$-eigenfunctions $\eta^l_\mu(x)$ (see (2.4)) in the limit as $q \to 1$.

There exists another, more complicated, one-parameter family of self-adjoint operators, closely related to H_1. They are defined as

$$H_1^{(\varphi)} := \frac{a}{2}\, q^{J_0} - b - q^{J_0/4}\, [\cos\varphi \cdot J_1 - \sin\varphi \cdot J_2]\, q^{J_0/4},$$

where $0 \le \varphi < 2\pi$, $J_\pm = J_1 \pm {\rm i}J_2$ and a, b are the same as in (5.1). These operators are bounded and self-adjoint. Repeating the same reasoning, as for the operator H_1, we arrive at the following theorem.

Theorem 3. *The eigenfunctions of the operator $H_1^{(\varphi)}$ are*

$$\xi^l_\lambda(e^{{\rm i}\varphi}x; q) = \frac{(q^{l/2}\, e^{{\rm i}\varphi}x; q)_\infty\, (q^{(l+1)/2}\, e^{{\rm i}\varphi}x; q)_\infty}{(q^{(1-2l)/4}\, e^{{\rm i}(\theta+\varphi)}x; q)_\infty\, (q^{(1-2l)/4}\, e^{-{\rm i}(\theta-\varphi)}x; q)_\infty}\,,$$

where $\lambda = (1-\cos\theta)/(q^{-1/2} - q^{1/2})$. Its spectrum is simple and covers the interval $[0, c]$, where $c = 2q^{1/2}/(1-q)$. The corresponding eigenvalues λ are the same as for H_1.

The operators $H_1^{(\varphi)}$ are q-extensions of $\mathrm{su}_{1,1}$-operators $J_0^{\rm cl} - \cos\varphi\, J_1^{\rm cl} + \sin\varphi\, J_2^{\rm cl}$.

5. More Hamiltonians with bounded continuous spectra

This section deals with eigenfunctions $\xi^l_\lambda(x;\varphi)$ and eigenvalues of a one-parameter family of the self-adjoint operators

$$H^{(\varphi)} := \frac{1}{2}(q^{1/4}J_+ + q^{-1/4}J_-)\, q^{J_0/2} + \frac{\cos\varphi}{q^{-1/2} - q^{1/2}}\, q^{J_0} \tag{6.1}$$

of the representation T_l^+ of the algebra $\mathrm{su}_q(1,1)$: $H^{(\varphi)}\, \xi^l_\lambda(x;\varphi) = \lambda\, \xi^l_\lambda(x;\varphi)$. Using the relations (2.11)–(2.13) we find that

$$H^{(\varphi)} f(x) = c(x^{-1} - 2q^{(2l+1)/4}\, \cos\varphi + q^{l+1/2}x) f(qx) - c(x^{-1} + q^{1/2-l}x) f(x),$$

where $c = (q^{(2l-1)/4})/2(q^{1/2} - q^{-1/2})$. By using this expression we find (details are given in [15]), that the eigenfunctions of $H^{(\varphi)}$ are

$$\xi^l_\lambda(x;\varphi) = \frac{(axe^{i\varphi};q)_\infty (axe^{-i\varphi};q)_\infty}{(bxe^{i(\theta-\varphi)};q)_\infty (bxe^{i(\varphi-\theta)};q)_\infty}\,, \qquad \lambda = \frac{\cos(\theta-\varphi)}{q^{-1/2} - q^{1/2}}\,,$$

where $a = q^{(2l+1)/4}$ and $b = q^{(1-2l)/4}$. A relation between the eigenfunctions $\xi^l_\lambda(x;\varphi)$ and the basis functions $f^l_n(x)$ is now an easy consequence of the generating function

$$\frac{(ae^{i\varphi}t;q)_\infty (ae^{-i\varphi}t;q)_\infty}{(e^{i(\theta-\varphi)}t;q)_\infty (e^{i(\varphi-\theta)}t;q)_\infty} = \sum_0^\infty P_n\left(\cos(\theta-\varphi);a|q\right)\, t^n$$

for the q-Meixner–Pollaczek polynomials $P_n(y;a|q)$, defined (see [14], section 3.9) as

$$P_n(\cos(\theta+\varphi);a|q) = a^{-n}\, e^{-in\varphi} \frac{(a^2;q)_n}{(q;q)_n}\, {}_3\phi_2\left(q^{-n}, ae^{i(\theta+2\varphi)}, ae^{-i\theta}\; a^2, 0\,; q, q\right).$$

Thus

$$\xi^l_\lambda(x;\varphi) = \sum_{n=0}^\infty \frac{q^{(1-2l)n/4}}{c^\nu_n} P_n(\cos(\theta-\varphi);q^l|q) f^l_n(x).$$

To find a spectrum of the operator $H^{(\varphi)}$, we note that the q-Meixner–Pollaczek polynomials $P_n(\cos(\theta-\varphi)) \equiv P_n\left(\cos(\theta-\varphi);q^l|q\right)$ are orthogonal and the orthogonality relation has the form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_m(\cos(\theta-\varphi))P_n(\cos(\theta-\varphi))w_\varphi(\cos(\theta-\varphi))d\theta = \frac{(q^{2l};q)_n}{(q;q)_n}\delta_{mn},$$

where

$$w_\varphi(\cos(\theta-\varphi)) = (q;q)_\infty (q^{2l};q)_\infty \left| \frac{(e^{2\mathrm{i}(\theta-\varphi)};q)_\infty}{q^l(e^{\mathrm{i}(\theta-\varphi)};q)_\infty (q^l e^{\mathrm{i}\theta};q)_\infty} \right|^2$$

(see formula (3.9.2) in [14]). This orthogonality relation can be written as

$$\int_a^b \frac{(q;q)_n}{(q^{2l};q)_n} P_m(\lambda(q^{-1/2}-q^{1/2}))P_n(\lambda(q^{-1/2}-q^{1/2})))\hat{w}(\lambda)d\lambda = \delta_{mn},$$

where

$$\hat{w}(\lambda) = \frac{w_\varphi(\lambda(q^{-1/2}-q^{1/2}))(q^{-1/2}-q^{1/2})}{\sin(\varphi-\theta)}, \quad a = \frac{-\cos(\pi+\varphi)}{q^{-1/2}-q^{1/2}}, \quad b = \frac{\cos(\pi-\varphi)}{q^{-1/2}-q^{1/2}}. \tag{6.2}$$

Therefore, we may formulate the following theorem:

Theorem 4. *The operator $H^{(\varphi)}$ has continuous and simple spectrum, which completely covers the interval (a,b), where a and b are given by (6.2).*

Continuity of the spectrum means that the eigenfunctions $\xi^l_\lambda(x;\varphi)$ do not belong to the Hilbert space $\mathcal{H}_l$. They belong to the space of functionals on the linear space $\mathcal{L}_l$, which can be considered as a space of generalized functions on $\mathcal{L}_l$.

The classical limit (that is, the limit $q \to 1$) has sense only for the operator

$$H^{(\pi/2)} = \tfrac{1}{2}(q^{1/4}J_+ + q^{-1/4}J_-)q^{J_3/2}.$$

When $q \to 1$ the operator $H^{(\pi/2)}$ tends to the operator $J_1^{(\mathrm{cl})}$: $\lim_{q\to 1} H^{(\pi/2)} = J_1^{(\mathrm{cl})}$. In this limit the basis elements (2.6) turn into the basis elements $g^l_n(x)$ of the representation space for the Lie algebra $\mathrm{su}(1,1)$ and the eigenfunctions $\xi^l_\lambda(x;\pi/2)$ of $H^{(\pi/2)}$ into the eigenfunctions

$$\xi^l_\lambda(x) := (1+ix)^{-l-i\lambda}(1-ix)^{-l+i\lambda}$$

of the operator $J_1^{(\mathrm{cl})}$. They are related to the eigenfunctions of the operator $J_0^{(\mathrm{cl})}$ as

$$\xi_\lambda^l(x) = \sum_{n=0}^\infty \left(\frac{n!}{(2l)_n} \right)^{1/2} P_n^{(l)}(\lambda; \pi/2)\, f_n^l(x) \equiv \sum_{n=0}^\infty P_n^{(l)}(\lambda; \pi/2)\, x^n \,,$$

where $P_n^{(l)}(\lambda; \pi/2)$ are the classical Meixner–Pollazcek polynomials, defined by the formula

$$P_n^{(l)}(x;\varphi) := \frac{(2l)_n}{n!}\, e^{in\varphi}\, {}_2F_1(-n,\, \nu + ix;\, 2l;\, 1 - e^{-2i\varphi}), \quad l > 0, \quad 0 < \varphi < \pi.$$

The Meixner–Pollazcek polynomials in the expression for $\xi_\lambda^l(x)$ are a limit case of the corresponding q-Meixner–Pollazcek polynomials (see [14], section 5.9).

6. Hamiltonians with bounded discrete spectra

In this section we consider the operator

$$H_2 = q^{3J_0/4}(J_+ + J_-)q^{3J_0/4} - \left([J_0 - l]_q\, q^{l/2} + [J_0 + l]_q\, q^{-l/2} \right) q^{3J_0/2}$$

(note that this operator depends on the index l of the representation T_l^+). It acts on the basis elements (2.6) by the formula

$$H_2\, f_k^l = a_{k+1} f_{k+1}^l + a_k f_{k-1}^l - q^{3(l+k)/2} d_k f_k^l, \tag{7.1}$$

where

$$a_k = q^{3(l+k)/2 - 3/4} \sqrt{[k]_q\, [2l + k - 1]_q}, \quad d_k = [k]_q\, q^{(l-1)/2} + [2l + k]_q\, q^{-(l-1)/2}.$$

By using this action it is easy to check that H_2 is bounded self-adjoint operator. We look for eigenfunctions of the operator H_2 in the form

$$\chi_\lambda^l(x) = \sum_{k=0}^\infty P_k(\lambda) f_k^l(x)\,.$$

The equation

$$H_2\, \chi_\lambda^l(x) = \sum_{k=0}^\infty P_k(\lambda)\, H_2 f_k^l(x) = \lambda \sum_{k=0}^\infty P_k(\lambda)\, f_k^l(x)$$

and the formula (7.1) lead to the following recurrence relation for the polynomials $P_k(\lambda)$, which after simple transformations can be written as

$$-q^{k+l}\, [(1 - q^{k+1})(1 - q^{2l+k})]^{1/2} P_{k+1}(\lambda) - q^{k+l-1}\, [(1 - q^k)(1 - q^{2l+k-1})]^{1/2} P_{k-1}(\lambda)$$
$$+ (q^k - q^{2k+2l} + q^{2l+k-1} - q^{2k+2l-1})\, P_k(\lambda) = (1 - q^{-1})\, \lambda\, P_k(\lambda). \tag{7.2}$$

Upon making the substitution

$$P_k(\lambda) = [(q^{2l}; q)_k / (q; q)_k]^{1/2}\, q^{-lk}\, P'_k(\lambda)$$

in this recurrence relation, one derives the equation

$$-q^k\, (1 - q^{2l+k})\, P'_{k+1}(\lambda) - q^{k+l-1}\, (1 - q^k)\, P'_{k-1}(\lambda)$$

$$+ (q^k - q^{2k+2l} + q^{2l+k-1} - q^{2k+2l-1})\, P'_k(\lambda) = (1-q^{-1})\,\lambda\, P'_k(\lambda).$$

This is the recurrence relation for the little q-Laguerre (Wall) polynomials

$$p_k(q^y; q^{2l-1}|q) = {}_2\phi_1(q^{-k}, 0;\; q^{2l};\; q; q^{y+1}) = (q^{1-2l-k};q)_k^{-1}\; {}_2\phi_0(q^{-k}, q^{-y};\; -;\; q; q^{y-2l+1})$$

with $q^y = (1-q^{-1})\lambda$. Thus, we have $P'_k(\lambda) = p_k(q^y; q^{2l-1}|q)$, $q^y = (1-q^{-1})\lambda$, and, consequently,

$$P_k(\lambda) = [(q^{2l};q)_k/(q;q)_k]^{1/2} q^{-lk}\, p_k(q^y; q^{2l-1}|q). \tag{7.3}$$

This means that eigenfunctions of the operator H_2 are of the form

$$\chi^l_\lambda(x) = \sum_{k=0}^{\infty} q^{(1-6l)k/4}\, \frac{(q^{2l};q)_k}{(q;q)_k}\, p_k(q^y; q^{2l-1}|q)\, x^k, \qquad q^y = (1-q^{-1})\lambda.$$

To sum up the right-hand side of this relation, one needs to know a generating function

$$F(x;\, t;\, a|q) := \sum_{n=0}^{\infty} \frac{(aq;q)_n}{(q;q)_n}\, p_n(x;\, a|q)\, t^n \tag{7.4}$$

for the little q-Laguerre polynomials

$$p_n(x;\, a|q) := {}_2\phi_1(q^{-n}, 0;\; aq;\; q; qx) = (a^{-1}q^{-n};q)_n^{-1}, {}_2\phi_0(q^{-n}, x^{-1};\; -;\; q; x/a). \tag{7.5}$$

To evaluate (7.4), we start with the second expression in (7.5) in terms of the basic hypergeometric series ${}_2\phi_0$. Substituting it into (7.4) and using the relation

$$(q^{-n};q)_k/(q;q)_k = (-1)^k q^{-kn+k(k-1)/2}(q;q)_n/(q;q)_k(q;q)_{n-k},$$

one obtains that

$$F(x;\, t;\, a|q) = \sum_{n=0}^{\infty} (-aqt)^n q^{n(n-1)/2} \sum_{k=0}^{n} \frac{(x^{-1};q)_k}{(q;q)_k(q;q)_{n-k}} (q^{-n}x/a)^k. \tag{7.6}$$

Interchanging the order of summations in (7.6) leads to the desired expression

$$F(x;\, t;\, a|q) = E_q(-aqt)\, {}_2\phi_0(x^{-1}, 0;\; -;\; q; xt), \tag{7.7}$$

where $E_q(z) = (-z;q)_\infty$ is the q-exponential function of Jackson.

Similarly, if one substitutes into (7.4) the explicit form of the little q-Laguerre polynomials in terms of ${}_2\phi_1$ from (7.5), this yields an expression

$$F(x;\, t;\, a|q) = \frac{E_q(-aqt)}{E_q(-t)}\, {}_2\phi_1(0, 0;\; q/t;\; q; qx). \tag{7.8}$$

Using the explicit form of the generating function (7.7) for the little q-Laguerre polynomials, we arrive at

$$\chi^l_\lambda(x) = (q^{(2l+1)/4}x; q)_\infty\, {}_2\phi_0(q^{-y},\, 0;\, -\,;\; q;\; q^{y-(6l-1)/4}x), \tag{7.9}$$

where, as before, $q^y = (1-q^{-1})\lambda$. Another expression for $\chi^l_\lambda(x)$ can be written by using formula (7.8).

Due to the orthogonality relation

$$(q^{2l};q)_\infty \sum_{k=0}^{\infty} \frac{q^{2lk}}{(q;q)_k}\, p_m(q^k;q^{2l-1}|q)\, p_n(q^k;q^{2l-1}|q) = \frac{q^{2ln}(q;q)_n}{(q^{2l};q)_n}\, \delta_{mn} \tag{7.10}$$

for little q-Laguerre polynomials (see formula (3.20.2) in [14]), spectrum of the operator H_2 coincides with the set of points $q^n/(1-q^{-1})$, $n = 0,1,2,\cdots$. This means that the eigenfunctions

$$\chi^l_{\lambda_n}(x), \qquad n = 0,1,2,\cdots, \qquad \lambda_n = q^n/(1-q^{-1}), \tag{7.11}$$

constitute a basis in the representation space. We thus proved the following theorem.

Theorem 5. *The operator H_2 has a simple discrete spectrum, which consists of the points $q^n/(1-q^{-1})$, $n = 0,1,2,\cdots$. The corresponding eigenfunctions $\chi^l_{\lambda_n}$ form an orthogonal basis in the space $\mathcal{H}_l$.*

We conclude this section by calling attention to the circumstance that operators with bounded discrete spectra, representable by a Jacobi matrix, are important for studying families of q-orthogonal polynomials and their dual properties. A detailed discussion of several examples of such type can be found in [16].

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