SOLUTION OF THE PRINCIPAL CHIRAL FIELD PROBLEM AS "MATHEMATICA" ALGORITHM

M.A. MUKHTAROV
Institute of Mathematics and Mechanics
370602, Baku, F.Agaev str. 9, Azerbaijan

Solutions of the principal chiral field problem are constructed by means of Mathematica software.

1. The problem of constructing of the solutions of self-dual Yang-Mills (SDYM) model and its dimensional reductions, the principal chiral field problem in our case, in the explicit form for semisimple Lie algebra, rank of which is greater than two, remains important for the present time. The interest arises from the fact that almost all integrable models in one, two and (1+2)-dimensions are symmetry reductions of SDYM or they can be obtained from it by imposing the constraints on Yang-Mills potentials [1-12].

This work is a direct continuation of [13-15], where the exact solutions of the principal chiral field problem have been derived, and it shows how to obtain the further results using determined Mathematica algorithm. The discrete symmetry transformation method [12] applied here allows to generate new solutions from the old ones in much more easier way than applying methods from [11], and the case of SL(3,C) algebra gives us a key to construct solutions for an arbitrary semisimple algebra.

2. Equations of the principal chiral field problem are the systems of equations for the element f, taking values in the semisimple algebra,

$$\left(\theta_i-\theta_j\right)\frac{\partial^2 f}{\partial x_i \partial x_j}=\left[\frac{\partial f}{\partial x_i},\frac{\partial f}{\partial x_j}\right] \tag{1}$$

In the case of two-dimensional space: $\theta_1=1$, $\theta_2=-1$, $x_1=\xi$, $x_2=\nu$.

Following [12], for the case of a semisimple Lie algebra and for an element f being a solution of (1), the following statement takes place:

There exists such an element S taking values in a gauge group that

$$S^{-1}\frac{\partial S}{\partial x_i}=\frac{1}{\tilde{f}_-}\left[\frac{\partial \tilde{f}}{\partial x_i},X_M^+\right]-\theta_i\frac{\partial}{\partial x_i}\frac{1}{\tilde{f}_-}X_M^+ \tag{2}$$

Here X_M^+ is the element of the algebra corresponding to its maximal root divided by its norm, i.e.,

$$\left[X_M^+,\ X^-\right]=H,\ \left[H,\ X^\pm\right]=\pm 2X^\pm \quad ,$$

$-\ \tilde{f}_-$ - is the coefficient function in the decomposition of $\tilde{f}$ of the element corresponding to the minimal root of the algebra, $\tilde{f}=\sigma f\sigma^{-1}$ and where σ is an automorfism of the algebra, changing the positive and negative roots.

In the case of algebra SL(3,C) we'll consider the case of three dimensional representation of algebra and the following form of $\sigma=\begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix}$.

The discrete symmetry transformation, producing new solutions from the known ones, is as follows:

$$\frac{\partial F}{\partial x_i}=S\frac{\partial \tilde{f}}{\partial x_i}S^{-1}+\theta_i\frac{\partial S}{\partial x_i}S^{-1} \tag{3}$$

3. Let's represent the explicit formulae for transformation in the case of SL(3,C) algebra

$$f=\alpha_1X_1^+ + \alpha_2X_2^+ + \alpha_{1,2}X_{1,2}^+ + \tau_1h_1 + \tau_2h_2 + a_1X_1^- + a_2X_2^- + a_{1,2}X_{1,2}^- \quad , \tag{4}$$

In connection with the general scheme, first of all, it is necessary to find the solution of the equations (2) for the SL(3,C) valued function S for given f, solution of equations (1).

From (2) it is clear that S is upper triangular matrix and can be represented in the following form:

$$S = \exp \beta_1 X_1^+ \exp \beta_{1,2} X_{1,2}^+ \exp \beta_2 X_2^+ \exp \beta_0 H \quad , \tag{5}$$

where $H=h_1+h_2$.

After substitution of the last representation of S into (2) and taking into account (4), we have at every step of the recurrent procedure the following relations

$$\beta_0 = \ln \alpha_{1,2} , \quad \beta_1 = \alpha_2 , \quad \beta_2 = \alpha_1$$
$$(\beta_{1,2})_{x_i} = (\alpha_{1,2})_{zx_i} - (\delta_1 + \delta_2)_{x_i} \alpha_{1,2} - (\alpha_1)_{x_i} \alpha_2 \tag{6}$$

As the initial solution we'll take the explicit solution f belonging to the algebra of upper triangular matrixes:

$$f = \alpha_1 X_1^+ + \alpha_2 X_2^+ + \alpha_{1,2} X_{1,2}^+ + \tau_1 h_1 + \tau_2 h_2 \tag{7}$$

The component form of self-duality equations for this case is following

$$\frac{\partial^2 \tau_\nu}{\partial x_i \partial x_j} = 0 ,$$
$$\frac{\partial^2 a_\nu}{\partial x_i \partial x_j} = \{\delta_\nu , \alpha_\nu\}_{x_i , x_j} \quad , \quad i=1,2, \tag{8}$$
$$\frac{\partial^2 a_{1,2}}{\partial x_i \partial x_j} = \{\delta_1 + \delta_2 , \alpha_{1,2}\}_{x_i , x_j} \quad ,$$

where $\delta_1 = 2\tau_1 - \tau_2$, $\delta_2 = 2\tau_2 - \tau_1$ and figure brackets of two functions g_1 and g_2 denotes :

$$\{g_1 , g_2\}_{x_i , x_j} = \frac{\partial g_1}{\partial x_i}\frac{\partial g_2}{\partial x_j} - \frac{\partial g_2}{\partial x_j}\frac{\partial g_1}{\partial x_i} .$$

The general solution of system (8) takes the form

$$\tau_i = \sum_{s=1} \tau_i^s (x_s) \; , \; \alpha_i = \oint_c \alpha_i(\lambda) \exp(-\bar{\delta}_i(\lambda)) d\lambda ,$$
$$\bar{\delta}_i(\lambda) = \sum_{s=1} \frac{\tau_i^s (x_s)}{\lambda + \theta_s} ,$$
$$\alpha_{1,2} = \oint_c \alpha_{1,2}(\lambda) \exp(-\bar{\delta}_1(\lambda) - \bar{\delta}_2(\lambda)) d\lambda +$$
$$+ \oint_c \alpha_1(\lambda) \exp(-\bar{\delta}_1(\lambda)) d\lambda \oint_c \frac{d\lambda' \alpha_2(\lambda') \exp(-\bar{\delta}_2(\lambda'))}{\lambda - \lambda'} \tag{9}$$

Here the circle integration goes over the complex parameter λ.

By the direct check one can be convinced that (9) are the solutions of equations (8). The formulae (9) can also be obtained as a solution of homogeneous Riemann problem in the case of the solvable algebra [11].

Let's represent two types of Backlund transformation by means of which one can construct new types of solutions of equations (8) from the known solution (9). For solutions of first two equations of (8) this two Backlund transformations are the same:

$$\theta_s (\alpha_i^k)_{x_s} - (\delta_i)_{x_s} \alpha_i^k = (\alpha_i^{k+1})_{x_s} \; , i = 1,2 \tag{10}$$

For solutions of the third equation of the system (8) they are different:

$$\theta_s (\alpha_{1,2}^{0,k})_{x_s} - (\delta_1 + \delta_2)_{x_s} \alpha_{1,2}^{0,k} - (\alpha_1^k)_{x_s} \alpha_2^k = (\alpha_{1,2}^{0,k+1})_{x_s} \tag{11}$$

and

$$\theta_s(\alpha_{1,2}^{k,0})_{x_s} - (\delta_1+\delta_2)_{x_s}\alpha_{1,2}^{k,0} - \alpha_1^k(\alpha_2^k)_{x_s} = (\alpha_{1,2}^{k+1,0})_{x_s} \tag{12}$$

Note that starting, zero step of upper transformations procedure coincides with initial solutions (9).
Let's return to the solution of the equation (7) at the first step of the recurrent procedure.
Comparing (6) and (12) we came to the conclusion that $\beta_{1,2} = \alpha_{1,2}^{0,1}$.
Finally, knowing all components of matrix S and using (3) we can express the solution

$$F = F_1^+X_1^+ + F_2^+X_2^+ + F_{1,2}^+X_{1,2}^+ + F_1^0h_1 + F_2^0h_2 + F_1^-X_1^- + F_2^-X_2^- + F_{1,2}^-X_{1,2}^-$$

of self-duality equations at the first step of the recurrent procedure in terms of chains (10)-(12):

$$\begin{gathered}
F_1^0 = \tau_1 + \frac{\alpha_{1,2}^{1,0}}{\alpha_{1,2}^{0,0}},\ F_2^0 = \tau_2 + \frac{\alpha_{1,2}^{0,1}}{\alpha_{1,2}^{0,0}} \\
F_{1,2}^- = \frac{1}{\alpha_{1,2}^{0,0}},\ F_1^- = \frac{\alpha_2^0}{\alpha_{1,2}^{0,0}},\ F_2^- = -\frac{\alpha_1^0}{\alpha_{1,2}^{0,0}} \\
F_1^+ = -\frac{1}{\alpha_{1,2}^{0,0}}\begin{vmatrix}\alpha_1^0 & \alpha_1^1 \\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{1,0}\end{vmatrix},\ F_2^+ = -\frac{1}{\alpha_{1,2}^{0,0}}\begin{vmatrix}\alpha_2^0 & \alpha_2^1 \\ \alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1}\end{vmatrix} \\
F_{1,2}^+ = \frac{1}{\alpha_{1,2}^{0,0}}\begin{vmatrix}\alpha_{1,2}^{0,0} & \alpha_{1,2}^{0,1} \\ \alpha_{1,2}^{1,0} & \alpha_{1,2}^{1,1}\end{vmatrix}
\end{gathered} \tag{13}$$

Using the equations of the principal chiral field problem for the group-valued element

$$\theta_i(g)_{x_i}g^{-1} = (f)_{x_i}$$

the relations (3) can be rewritten as

$$\theta_i(S_n\sigma g_n)_{x_i}(S_n\sigma g_n)^{-1} = (f_{n+1})_{x_i} \quad ,$$

So we see that the group valued elements g_{n+1} and g_n are connected by the relation

$$g_{n+1} = S_n\sigma g_n \tag{14}$$

Let's represent the explicit formulae of the recurrent procedure of obtaining the group-valued element solutions.
At every step, as it shown in [5], S is upper triangular matrix and can be represented in the following form:

$$S_n = exp(\beta_1)_nX_1^+\ exp(\beta_{1,2})_nX_{1,2}^+\ exp(\beta_2)_nX_2^+\ exp(\beta_0)_nH \quad , \tag{15}$$

where $H=h_1+h_2$ and for g_n we use the following parameterization:

$$\begin{aligned}
g_n = {} & exp(\eta_1^+)_nX_1^+\ exp(\eta_{1,2}^+)_nX_{1,2}^+\ exp(\eta_2^+)_nX_2^+\ exp((t_1)_nh_1 + (t_2)_nh_2)\times \\
& \times exp(\eta_2^-)_nX_2^-\ exp(\eta_{1,2}^-)_nX_{1,2}^-\ exp(\eta_1^-)_nX_1^-
\end{aligned} \tag{16}$$

with

$$g_0 = exp(\eta_1^+)_0X_1^+\ exp(\eta_{1,2}^+)_0X_{1,2}^+\ exp(\eta_2^+)_0X_2^+\ exp((t_1)_0h_1 + (t_2)_0h_2)$$

as an initial solution.

Hereafter, $X_1^\pm$, $X_2^\pm$, $X_{1,2}^\pm$, h_1, h_2 are the generators of SL(3,C) algebra.

Following the general scheme from [5] we have at
(0)-step:

$$(t_i)_0 = \tau_i^{-1} \equiv \nu_i \, , (\eta_i^+)_0 = \alpha_i^{-1}, i = 1,2 \, , (\eta_{1,2}^+)_0 = \alpha_{1,2}^{-1,0} \quad ;$$

The further calculation we deliver to Mathematica program:

$$(t_i)_0 = \tau_i^{-1} \equiv \nu_i \, , (\eta_i^+)_0 = \alpha_i^{-1}, i = 1,2 \, , (\eta_{1,2}^+)_0 = \alpha_{1,2}^{-1,0} \quad ;$$

The further calculation we deliver to Mathematica program:

$$X_1^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad X_2^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad X_{1,2}^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$X_1^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad X_2^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad X_{1,2}^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \quad w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix};$$

G_0 = MatrixExp[$n_1 X_1^+$].MatrixExp[$n_{1,2} X_{1,2}^+$].MatrixExp[$n_2 X_2^+$].MatrixExp[$r_1 h_1$].
MatrixExp[$r_2 h_2$];
S_1 = MatrixExp[$b_1 X_1^+$].MatrixExp[$b_{1,2} X_{1,2}^+$].MatrixExp[$b_2 X_2^+$].MatrixExp[$b_0 (h_1 + h_2)$];
$G_1 = S_1$.w.G_0;
MatrixForm[G_1]

Null^4

$$\begin{pmatrix} -e^{-b_0+r_1}(b_1 b_2 + b_{1,2}) & e^{-r_1+r_2} b_1 - e^{-b_0-r_1+r_2} n_1 (b_1 b_2 + b_{1,2}) & e^{b_0-r_2} + e^{-r_2} b_1 n_2 - e^{-b_0-r_2}(b_1 b_2 + b_{1,2})(n_1 n_2 + n_{1,2}) \\ -e^{-b_0+r_1} b_2 & e^{-r_1+r_2} - e^{-b_0-r_1+r_2} b_2 n_1 & e^{-r_2} n_2 - e^{-b_0-r_2} b_2 (n_1 n_2 + n_{1,2}) \\ -e^{-b_0+r_1} & -e^{-b_0-r_1+r_2} n_1 & -e^{-b_0-r_2}(n_1 n_2 + n_{1,2}) \end{pmatrix}$$

G_1 /. $(b_1 b_2 + b_{1,2}) \to a_{1,2}[1,0]$ /. $n_1 n_2 + n_{1,2} \to a_{1,2}[0,-1]$ /. $b_0 \to \text{Log}[a_{1,2}[0,0]]$ /.
$b_1 \to a_1[0]$
/. $b_2 \to a_2[0]$ /. $n_1 \to a_1[-1]$ /. $n_2 \to a_2[-1]$

$$\{\{-e^{-\text{Log}[a_{1,2}[0,0]]+r_1} a_{1,2}[1,0], e^{-r_1+r_2} a_1[0] - e^{-\text{Log}[a_{1,2}[0,0]]-r_1+r_2} a_1[-1] a_{1,2}[1,0],$$
$$e^{\text{Log}[a_{1,2}[0,0]]-r_2} + e^{-r_2} a_1[0] a_2[-1] - e^{-\text{Log}[a_{1,2}[0,0]]-r_2} a_{1,2}[0,-1] a_{1,2}[1,0]\},$$
$$\{-e^{-\text{Log}[a_{1,2}[0,0]]+r_1} a_2[0], e^{-r_1+r_2} - e^{-\text{Log}[a_{1,2}[0,0]]-r_1+r_2} a_1[-1] a_2[0],$$
$$e^{-r_2} a_2[-1] - e^{-\text{Log}[a_{1,2}[0,0]]-r_2} a_2[0] a_{1,2}[0,-1]\},$$
$$\{-e^{-\text{Log}[a_{1,2}[0,0]]+r_1}, -e^{-\text{Log}[a_{1,2}[0,0]]-r_1+r_2} a_1[-1], -e^{-\text{Log}[a_{1,2}[0,0]]-r_2} a_{1,2}[0,-1]\}\}$$

G_1 = FullSimplify[%]

$$\{\{-\frac{e^{r_1} a_{1,2}[1,0]}{a_{1,2}[0,0]}, \frac{e^{-r_1+r_2}(a_1[0] a_{1,2}[0,0] - a_1[-1] a_{1,2}[1,0])}{a_{1,2}[0,0]},$$
$$\frac{e^{-r_2}(a_1[0] a_2[-1] a_{1,2}[0,0] + a_{1,2}[0,0]^2 - a_{1,2}[0,-1] a_{1,2}[1,0])}{a_{1,2}[0,0]}\}, \{-\frac{e^{r_1} a_2[0]}{a_{1,2}[0,0]},$$
$$\frac{e^{-r_1+r_2}(-a_1[-1] a_2[0] + a_{1,2}[0,0])}{a_{1,2}[0,0]}, \frac{e^{-r_2}(-a_2[0] a_{1,2}[0,-1] + a_2[-1] a_{1,2}[0,0])}{a_{1,2}[0,0]}\},$$
$$\{-\frac{e^{r_1}}{a_{1,2}[0,0]}, -\frac{e^{-r_1+r_2} a_1[-1]}{a_{1,2}[0,0]}, -\frac{e^{-r_2} a_{1,2}[0,-1]}{a_{1,2}[0,0]}\}\}$$

G_1 = FullSimplify[% /. $a_1[0] a_2[-1] \to a_{1,2}[1,-1] - a_{1,2}[0,0]$
/. $a_1[-1] a_2[0] \to a_{1,2}[0,0] - a_{1,2}[-1,1]$]

$$\left\{\left\{-\frac{e^{r_1}\, a_{1,2}[1,0]}{a_{1,2}[0,0]}, \frac{e^{-r_1+r_2}\,(a_1[0]\, a_{1,2}[0,0] - a_1[-1]\, a_{1,2}[1,0])}{a_{1,2}[0,0]}, \frac{e^{-r_2}\,(a_{1,2}[0,0]\, a_{1,2}[1,-1] - a_{1,2}[0,-1]\, a_{1,2}[1,0])}{a_{1,2}[0,0]}\right\},\right.$$

$$\left\{-\frac{e^{r_1}\, a_2[0]}{a_{1,2}[0,0]}, \frac{e^{-r_1+r_2}\, a_{1,2}[-1,1]}{a_{1,2}[0,0]}, \frac{e^{-r_2}\,(-a_2[0]\, a_{1,2}[0,-1] + a_2[-1]\, a_{1,2}[0,0])}{a_{1,2}[0,0]}\right\},$$

$$\left.\left\{-\frac{e^{r_1}}{a_{1,2}[0,0]}, -\frac{e^{-r_1+r_2}\, a_1[-1]}{a_{1,2}[0,0]}, -\frac{e^{-r_2}\, a_{1,2}[0,-1]}{a_{1,2}[0,0]}\right\}\right\}$$

$T_2[1] = G_1[[3, 3]];$
$T_1[1] = \text{Minors}[G_1][[3, 3]];$
$\alpha_2[1] = G_1[[2, 3]]/T_2[1];$
$\beta_2[1] = G_1[[3, 2]]/T_2[1];$
$\beta_1[1] = \text{Minors}\,[G_1][[3, 2]]/T_1[1];$
$\alpha_1[1] = \text{Minors}\,[G_1][[2, 3]]/T_1[1];$
$\alpha_{1,2}[1] = -\text{Minors}\,[G_1][[1, 3]]/T_1[1];$
$\beta_{1,2}[1] = -\text{Minors}\,[G_1][[3, 1]]/T_1[1];$
$\text{FullSimplify}[T_2[1]]$

$$-\frac{e^{-r_2}\, a_{1,2}[0,-1]}{a_{1,2}[0,0]}$$

$\text{FullSimplify}[T_1[1]]$

$$\frac{e^{-r_1}\,(-(a_1[-1]\, a_2[0] + a_{1,2}[-1,1])\, a_{1,2}[0,-1] + a_1[-1]\, a_2[-1]\, a_{1,2}[0,0])}{a_{1,2}[0,0]^2}$$

$T_1[1] = \text{FullSimplify}[\%\ /.\ a_1[-1]\, a_2[0] \to a_{1,2}[0,0] - a_{1,2}[-1,1]$
$/.\ a_1[-1]\, a_2[-1] \to a_{1,2}[0,-1] - a_{1,2}[-1,0]]$

$$-\frac{e^{-r_1}\, a_{1,2}[-1,0]}{a_{1,2}[0,0]}$$

$\text{FullSimplify}[\beta_1[1]]$

$$\frac{e^{2 r_1 - r_2}\, a_2[-1]\, a_{1,2}[0,0]}{-(a_1[-1]\, a_2[0] + a_{1,2}[-1,1])\, a_{1,2}[0,-1] + a_1[-1]\, a_2[-1]\, a_{1,2}[0,0]}$$

$\beta_1[1] = \text{FullSimplify}[\%\ /.\ a_1[-1]\, a_2[0] \to a_{1,2}[0,0] - a_{1,2}[-1,1]$
$/.\ a_1[-1]\, a_2[-1] \to a_{1,2}[0,-1] - a_{1,2}[-1,0]]$

$$-\frac{e^{2 r_1 - r_2}\, a_2[-1]}{a_{1,2}[-1,0]}$$

$\text{FullSimplify}[\beta_{1,2}[1]]$

$$\frac{e^{r_1+r_2}\,(a_1[-1]\, a_2[0] + a_{1,2}[-1,1])}{(a_1[-1]\, a_2[0] + a_{1,2}[-1,1])\, a_{1,2}[0,-1] - a_1[-1]\, a_2[-1]\, a_{1,2}[0,0]}$$

$\beta_{1,2}[1] = \text{FullSimplify}[\%\ /.\ a_1[-1]\, a_2[0] \to a_{1,2}[0,0] - a_{1,2}[-1,1]$
$/.\ a_1[-1]\, a_2[-1] \to a_{1,2}[0,-1] - a_{1,2}[-1,0]]$

$$\frac{e^{r_1+r_2}}{a_{1,2}[-1,0]}$$

$\text{FullSimplify}[\beta_2[1]]$

$$\frac{e^{-r_1+2 r_2}\, a_1[-1]}{a_{1,2}[0,-1]}$$

$\text{FullSimplify}[\alpha_1[1]]$

$$\frac{a_{1,2}[0,0]\,(-a_1[0]\,a_{1,2}[0,-1]+a_1[-1]\,a_{1,2}[1,-1])}{-(a_1[-1]\,a_2[0]+a_{1,2}[-1,1])\,a_{1,2}[0,-1]+a_1[-1]\,a_2[-1]\,a_{1,2}[0,0]}$$

α_1[1] = FullSimplify[% /. a_1[-1] a_2[0] → $a_{1,2}$[0, 0] − $a_{1,2}$[-1, 1]

/. a_1[-1] a_2[-1] → $a_{1,2}$[0, -1] − $a_{1,2}$[-1, 0]]

$$\frac{a_1[0]\,a_{1,2}[0,-1]-a_1[-1]\,a_{1,2}[1,-1]}{a_{1,2}[-1,0]}$$

FullSimplify[α_2[1]]

$$a_2[0]-\frac{a_2[-1]\,a_{1,2}[0,0]}{a_{1,2}[0,-1]}$$

FullSimplify[$\alpha_{1,2}$[1]]

$$\frac{a_{1,2}[0,0]\,(a_1[0]\,(a_2[0]\,a_{1,2}[0,-1]-a_2[-1]\,a_{1,2}[0,0])+a_{1,2}[-1,1]\,a_{1,2}[1,-1])}{-(a_1[-1]\,a_2[0]+a_{1,2}[-1,1])\,a_{1,2}[0,-1]+a_1[-1]\,a_2[-1]\,a_{1,2}[0,0]}+a_{1,2}[1,0]$$

$\alpha_{1,2}$[1] = FullSimplify[% /. a_1[-1] a_2[0] → $a_{1,2}$[0, 0] − $a_{1,2}$[-1, 1]

/. a_1[-1] a_2[-1] → $a_{1,2}$[0, -1] − $a_{1,2}$[-1, 0]]

$$\frac{a_1[0]\,(-a_2[0]\,a_{1,2}[0,-1]+a_2[-1]\,a_{1,2}[0,0])-a_{1,2}[-1,1]\,a_{1,2}[1,-1]}{a_{1,2}[-1,0]}+a_{1,2}[1,0]$$

Expand[%]

$$-\frac{a_1[0]\,a_2[0]\,a_{1,2}[0,-1]}{a_{1,2}[-1,0]}+\frac{a_1[0]\,a_2[-1]\,a_{1,2}[0,0]}{a_{1,2}[-1,0]}-\frac{a_{1,2}[-1,1]\,a_{1,2}[1,-1]}{a_{1,2}[-1,0]}+a_{1,2}[1,0]$$

$\alpha_{1,2}$[1] = FullSimplify[% /. $a_{1,2}$[1, 0] → a_1[0] a_2[0] + $a_{1,2}$[0, 1] /

.$a_{1,2}$[1, -1] → a_1[0] a_2[-1] + $a_{1,2}$[0, 0] /. $a_{1,2}$[0, -1] → a_1[-1] a_2[-1] + $a_{1,2}$[-1, 0]]

$$\frac{1}{a_{1,2}[-1,0]}(-a_1[0]\,a_2[-1]\,(a_1[-1]\,a_2[0]+a_{1,2}[-1,1])+$$
$$(a_1[0]\,a_2[-1]-a_{1,2}[-1,1])\,a_{1,2}[0,0]+a_{1,2}[-1,0]\,a_{1,2}[0,1])$$

$\alpha_{1,2}$[1] = FullSimplify[% /. a_1[-1] a_2[0] → $a_{1,2}$[0, 0] − $a_{1,2}$[-1, 1]]

$$-\frac{a_{1,2}[-1,1]\,a_{1,2}[0,0]}{a_{1,2}[-1,0]}+a_{1,2}[0,1]$$

The second step.

$$B_1=\frac{-a_1[0]\,a_{1,2}[1,0]+a_1[1]\,a_{1,2}[0,0]}{a_{1,2}[0,0]};\quad B_2=\frac{-a_2[0]\,a_{1,2}[0,1]+a_2[1]\,a_{1,2}[0,0]}{a_{1,2}[0,0]};$$

$$B_0=\mathrm{Log}\Big[\frac{-a_{1,2}[1,1]\,a_{1,2}[0,0]+a_{1,2}[1,0]\,a_{1,2}[0,1]}{a_{1,2}[0,0]}\Big];$$

$$B_{1,2}=\frac{1}{(a_{1,2}[0,0])\hat{}2}\,a_{1,2}[0,0]\,(a_{1,2}[1,2]\,a_{1,2}[0,0]-a_{1,2}[1,1]\,a_{1,2}[0,1])\,a_{1,2}[0,1]$$

$$(a_{1,2}[1,1]\,a_{1,2}[0,0]-a_{1,2}[1,0]\,a_{1,2}[0,1]);$$

S_2 = MatrixExp[B_1 X_1^+].MatrixExp[$B_{1,2}$ $X_{1,2}^+$].MatrixExp[B_2 X_2^+].MatrixExp[B_0 (h_1 + h_2)];

G_2 = S_2.w.G_1; T_2[2] = G_2[[3, 3]]; T_1[2] = Minors[G_2][[3,3]];

α_2[2] = G_2[[2, 3]]/T_2[2]; β_2[2] = G_2[[3, 2]]/T_2[2]; β_1[2] = Minors [G_2][[3, 2]]/T_1[2];

α_1[2] = Minors [G_2][[2, 3]]/T_1[2]; $\alpha_{1,2}$[2] = −Minors [G_2][[1, 3]]/T_1[2];

$\beta_{1,2}$[2] = −Minors [G_2][[3, 1]]/T_1[2] FullSimplify[T_2[2]]

$$\frac{e^{-r_2}\,(a_{1,2}[0,0]\,a_{1,2}[1,-1]-a_{1,2}[0,-1]\,a_{1,2}[1,0])}{-a_{1,2}[0,1]\,a_{1,2}[1,0]+a_{1,2}[0,0]\,a_{1,2}[1,1]}$$

FullSimplify[T_1[2]]

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```
(e^-r1 (a_{1,2}[0, 0] (a_1[0] (a_2[0] a_{1,2}[0, -1] - a_2[-1] a_{1,2}[0, 0]) + a_{1,2}[-1, 1] a_{1,2}[1, -1]) -
     ((a_1[-1] a_2[0] + a_{1,2}[-1, 1]) a_{1,2}[0, -1] - a_1[-1] a_2[-1] a_{1,2}[0, 0]) a_{1,2}[1, 0])) /
  (a_{1,2}[0, 0] (-a_{1,2}[0, 1] a_{1,2}[1, 0] + a_{1,2}[0, 0] a_{1,2}[1, 1]))
T_1[1] = FullSimplify[%44 /. a_{1,2}[1, -1] → a_{1,2}[0, 0] + a_1[0] a_2[-1]
     /. a_{1,2}[0, -1] → a_{1,2}[-1, 0] + a_1[-1] a_2[-1] /. a_1[-1] a_2[0] → a_{1,2}[0, 0] - a_{1,2}[-1, 1]]
(e^-r1 (a_1[-1] a_1[0] a_2[-1] a_2[0] + a_1[0] (a_2[0] a_{1,2}[-1, 0] + a_2[-1] (a_{1,2}[-1, 1] - a_{1,2}[0, 0])) +
     a_{1,2}[-1, 1] a_{1,2}[0, 0] - a_{1,2}[-1, 0] a_{1,2}[1, 0])) / (-a_{1,2}[0, 1] a_{1,2}[1, 0] + a_{1,2}[0, 0] a_{1,2}[1, 1])
ch1 = (a_1[-1] a_1[0] a_2[-1] a_2[0] + a_1[0] (a_2[0] a_{1,2}[-1, 0] + a_2[-1] (a_{1,2}[-1, 1] - a_{1,2}[0, 0])) +
     a_{1,2}[-1, 1] a_{1,2}[0, 0] - a_{1,2}[-1, 0] a_{1,2}[1, 0]);
FullSimplify[ch1 /. a_{1,2}[1, 0] → a_{1,2}[0, 1] + a_1[0] a_2[0] /. a_1[-1] a_2[0] → a_{1,2}[0, 0] - a_{1,2}[-1, 1]]
a_{1,2}[-1, 1] a_{1,2}[0, 0] - a_{1,2}[-1, 0] a_{1,2}[0, 1]
T_2[2] =
  e^-r1 (a_{1,2}[-1, 1] a_{1,2}[0, 0] - a_{1,2}[-1, 0] a_{1,2}[0, 1]) /
     (-a_{1,2}[0, 1] a_{1,2}[1, 0] + a_{1,2}[0, 0] a_{1,2}[1, 1])
e^-r1 (a_{1,2}[-1, 1] a_{1,2}[0, 0] - a_{1,2}[-1, 0] a_{1,2}[0, 1])
-------------------------------------------------------------
   -a_{1,2}[0, 1] a_{1,2}[1, 0] + a_{1,2}[0, 0] a_{1,2}[1, 1]
FullSimplify[β_1[2]]
(e^(2 r1 - r2) a_{1,2}[0, 0] (a_2[0] a_{1,2}[1, -1] - a_2[-1] a_{1,2}[1, 0])) /
  (a_{1,2}[0, 0] (a_1[0] (-a_2[0] a_{1,2}[0, -1] + a_2[-1] a_{1,2}[0, 0]) - a_{1,2}[-1, 1] a_{1,2}[1, -1]) +
    ((a_1[-1] a_2[0] + a_{1,2}[-1, 1]) a_{1,2}[0, -1] - a_1[-1] a_2[-1] a_{1,2}[0, 0]) a_{1,2}[1, 0])
Zn1 = (a_{1,2}[0, 0] (a_1[0] (-a_2[0] a_{1,2}[0, -1] + a_2[-1] a_{1,2}[0, 0]) - a_{1,2}[-1, 1] a_{1,2}[1, -1]) +
     ((a_1[-1] a_2[0] + a_{1,2}[-1, 1]) a_{1,2}[0, -1] - a_1[-1] a_2[-1] a_{1,2}[0, 0]) a_{1,2}[1, 0]);
FullSimplify[
  Zn1 /. a_{1,2}[1, -1] → a_{1,2}[0, 0] + a_1[0] a_2[-1]
      /. a_{1,2}[0, -1] → a_{1,2}[-1, 0] + a_1[-1] a_2[-1] /. a_1[-1] a_2[0] → a_{1,2}[0, 0] - a_{1,2}[-1, 1] /.
   a_{1,2}[1, 0] → a_{1,2}[0, 1] + a_1[0] a_2[0]]
a_{1,2}[0, 0] (-a_1[0] a_2[-1] (a_1[-1] a_2[0] + a_{1,2}[-1, 1]) +
    (a_1[0] a_2[-1] - a_{1,2}[-1, 1]) a_{1,2}[0, 0] + a_{1,2}[-1, 0] a_{1,2}[0, 1])
FullSimplify[% /. a_1[-1] a_2[0] → a_{1,2}[0, 0] - a_{1,2}[-1, 1]]
a_{1,2}[0, 0] (-a_{1,2}[-1, 1] a_{1,2}[0, 0] + a_{1,2}[-1, 0] a_{1,2}[0, 1])
β_1[2] = e^(2 r1 - r2) (a_2[0] a_{1,2}[1, -1] - a_2[-1] a_{1,2}[1, 0]) /
     (-a_{1,2}[-1, 1] a_{1,2}[0, 0] + a_{1,2}[-1, 0] a_{1,2}[0, 1]);

FullSimplify[β_2[2]]

e^(-r1 + 2 r2) (a_1[0] a_{1,2}[0, 0] - a_1[-1] a_{1,2}[1, 0])
-------------------------------------------------------------
 a_{1,2}[0, 0] a_{1,2}[1, -1] - a_{1,2}[0, -1] a_{1,2}[1, 0]
FullSimplify[β_{1,2}[2]]
(e^(r1 + r2) (a_1[0] a_2[0] a_{1,2}[0, 0] - (a_1[-1] a_2[0] + a_{1,2}[-1, 1]) a_{1,2}[1, 0])) /
  (a_{1,2}[0, 0] (a_1[0] (a_2[0] a_{1,2}[0, -1] - a_2[-1] a_{1,2}[0, 0]) + a_{1,2}[-1, 1] a_{1,2}[1, -1]) -
    ((a_1[-1] a_2[0] + a_{1,2}[-1, 1]) a_{1,2}[0, -1] - a_1[-1] a_2[-1] a_{1,2}[0, 0]) a_{1,2}[1, 0])
FullSimplify[% /. a_{1,2}[1, -1] → a_{1,2}[0, 0] + a_1[0] a_2[-1]
     /. a_{1,2}[0, -1] → a_{1,2}[-1, 0] + a_1[-1] a_2[-1] /. a_{1,2}[1, 0] → a_{1,2}[0, 1] + a_1[0] a_2[0]
   /. a_1[-1] a_2[0] → a_{1,2}[0, 0] - a_{1,2}[-1, 1]]
(e^(r1 + r2) a_{1,2}[0, 1]) / (-a_1[0] a_2[-1] (a_1[-1] a_2[0] + a_{1,2}[-1, 1]) +
    (a_1[0] a_2[-1] - a_{1,2}[-1, 1]) a_{1,2}[0, 0] + a_{1,2}[-1, 0] a_{1,2}[0, 1])
```

FullSimplify[% /. a₁[−1] a₂[0] → a₁,₂[0, 0] − a₁,₂[−1, 1]]

$$\frac{e^{r_1+r_2}\, a_{1,2}[0,1]}{-a_{1,2}[-1,1]\, a_{1,2}[0,0]+a_{1,2}[-1,0]\, a_{1,2}[0,1]}$$

$$\beta_{1,2}[2]=\frac{e^{r_1+r_2}\, a_{1,2}[0,1]}{-a_{1,2}[-1,1]\, a_{1,2}[0,0]+a_{1,2}[-1,0]\, a_{1,2}[0,1]}$$

$$\frac{e^{r_1+r_2}\, a_{1,2}[0,1]}{-a_{1,2}[-1,1]\, a_{1,2}[0,0]+a_{1,2}[-1,0]\, a_{1,2}[0,1]}$$

At that point we stop the procedure, but it is obvious that it can easily be continued for any step and the resulting formulas correspond to those from [15].

As it is seen from formulas (11-12) for algebras of the rank higher than two, the number of corresponding Backlund transformations of the initial problem solutions will be equal to the rank of the algebra. Thus, it is necessary only to overcome the routine calculations using Mathematica 4-0 software.

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M.A. Muxtarov

ƏSAS KİRAL SAHƏ MƏSƏLƏSİNİN «MATEMATİKA» ALQORİTMİ KİMİ HƏLLİ

Əsas kiral sahə tənliyinin həlli «Matematika» proqramı alqoritminin köməyi ilə qurulmuşdur.

М.А. Мухтаров

РЕШЕНИЕ ЗАДАЧИ ГЛАВНОГО КИРАЛЬНОГО ПОЛЯ КАК "МАТЕМАТИКА" АЛГОРИТМ

Решения уравнений главного кирального поля построены посредством алгоритма на программе Математика.

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