LAX PAIR PRESENTATION OF WZNW MODEL

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One dimensional WZNW model obtained as a reduction of self-dual Yang-Mills equations has been presented in the normal Lax pair form.

1. The problem of constructing of the solutions of self-dual Yang-Mills (SDYM) model and its dimensional reductions, the one dimensional WZNW model in our case, in the explicit form for arbitrary semisimple Lie algebra, rank of which is greater than two, remains important for the present time. The interest arises from the fact that almost all integrable models in one, two and (1+2)-dimensions are symmetry reductions of SDYM or they can be obtained from it by imposing the constraints on Yang-Mills potentials [1-12].

This work is a direct continuation of [11], where the exact solutions have been derived with the use of Riemann Hilbert Problem formalism. The aim is to apply to this problem the discrete symmetry transformation method [12] that allows generating new solutions from the old ones in much more easier way than applying methods from [11]. The Lax pair presentation of the model under consideration is the first step in this program that we hope will give us a key to construct solutions for an arbitrary semisimple algebra.

2. The one dimensional reduction of self duality equations obtained in [11] are the equations for the element *f*, taking values in the semisimple algebra,

$$\frac{\partial^2 f}{\partial r^2}+2\frac{\partial f}{\partial r}-[H,[H,f]]-2[X^-,[X^+,f]]-2[X^+,[X^-,f]]+2[[\frac{\partial}{\partial r}-H,f],[X^+,f]]=0 \qquad (1)$$

Here $H, X^{\pm}$ are generators of $A_1(SL(2,C)$ algebra

$$\left[X_M^+,\ X^-\right]=H,\ \left[H,\ X^{\pm}\right]=\pm 2X^{\pm}$$

embedded to gauge algebra in the half-integer way.

Let's rewrite (1) in the equivalent form:

$$[\frac{1}{2}(\frac{\partial}{\partial r}+H)-[X^+,f],-\frac{1}{2}[\frac{\partial}{\partial r}-H,f]+X^-]-\frac{1}{2}[\frac{\partial}{\partial r}-H,f]+X^-=0$$

This equation after changing the variable $t=\ln r$ has the following form

$$[\frac{\partial}{\partial t}+\frac{1}{2}H-[X^+,f],-\frac{\partial f}{\partial t}+\frac{1}{2}[H,f]+X^-]-\frac{\partial f}{\partial t}+\frac{1}{2}[H,f]+X^-=0 \qquad (2)$$

Introducing the notation

$$\widetilde{F}=e^{\frac{1}{2}Ht}(-\frac{\partial f}{\partial t}+\frac{1}{2}[H,f]+X^-])e^{-\frac{1}{2}Ht}, \qquad (3)$$

multiplying (2) from the left side by $e^{\frac{1}{2}Ht}$ and from the right side by $e^{-\frac{1}{2}Ht}$, we obtain

$$\frac{\partial\widetilde{F}}{\partial t}-[[e^{\frac{1}{2}Ht}X^+e^{-\frac{1}{2}Ht},e^{\frac{1}{2}Ht}fe^{-\frac{1}{2}Ht}],\widetilde{F}]+\widetilde{F}=0$$

Due to the evident equality

$$e^{\frac{1}{2}Ht}X^+e^{-\frac{1}{2}Ht}=e^tX^+$$

the last equation can be rewritten in a form

$$\frac{\partial\widetilde{F}}{\partial t}-e^t[[X^+,\widetilde{f}],\widetilde{F}]+\widetilde{F}=0, \qquad (4)$$

where $\widetilde{f}=e^{\frac{1}{2}Ht}fe^{-\frac{1}{2}Ht}$.

In terms of these notations we have from (3) the following expression

$$\widetilde{F}=-\frac{\partial\widetilde{f}}{\partial t}+[H,\widetilde{f}]+X^-e^{-t}=0$$

Let's introduce the notation

$$F=e^t\widetilde{F}=-e^t\frac{\partial\widetilde{f}}{\partial t}+e^t[H,\widetilde{f}]+X^-=0$$

Then (4) has a form

$$\frac{\partial F}{\partial t}+[A,F]=0, \qquad (5)$$

where $A=-e^t[X^+,\widetilde{f}]$.

The equation (5) is one-dimensional evolution equation defined by Lax pair operators and it is one of the principal criteria of equations integrability.

3. From the presentation (5) it is followed that

$$\frac{\partial}{\partial t} spF^n = 0, \ for \ \forall n$$

and solution of the equations can be found in a form

$$F = \varphi F_0 \varphi^{-1}, \qquad (6)$$

where $\varphi(t)$ takes values in the corresponding Lie group and $F_0 = F|_{t=0}$.

From equation (5) and presentation (6) it is directly followed the expression for the operator A:

$$A = \varphi^{/} \varphi^{-1} \qquad (\varphi^{/} = \frac{\partial \varphi}{\partial t}) \qquad (7)$$

Let's consider the commutator of F with X^+:

$$[X^+, F] = [X^+, X^-] - e^t \frac{\partial}{\partial t}[X^+, \tilde{f}] + e^t [X^+, [H, \tilde{f}]] =$$

$$= H - e^t \frac{\partial}{\partial t}[X^+, \tilde{f}] - 2e^t [X^+, \tilde{f}] + e^t [X^+, [H, \tilde{f}]] =$$

$$= H - \frac{\partial}{\partial t}(e^t [X^+, \tilde{f}]) - e^t [X^+, \tilde{f}] + [H, e^t [X^+, \tilde{f}]].$$

Taking into account (6) and (7) the last expression can rewritten in a form

$$[X^+, \varphi F_0 \varphi^{-1}] = H - (\varphi^{/} \varphi^{-1})^{/} - \varphi^{/} \varphi^{-1} + [H, \varphi^{/} \varphi^{-1}].$$

Making the substitution $\varphi = e^{Ht} q$ and introducing a new variable $\tau = e^{-t}$, we have another form of equation (1)

$$\frac{\partial}{\partial \tau}(\frac{\partial q}{\partial \tau} q^{-1}) = [q F_0 q^{-1}, X^+] \qquad (8)$$

Equation (8) is one-dimensional WZNW (Wess-Zumino-Novikov-Witten) equation [16-18] and its partial exact solutions are the subject of further publications.

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WZNW MODELİNİN LAKSA CÜTÜ HALINDA TƏQDİM OLUNMASI

Avtodual Yanq-Mills tənliklərinin reduksiyasından alınan birölçülü WZNW modeli Laksa cütünün normal təqdim olunmasına gətirilib.

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ПРЕДСТАВЛЕНИЕ МОДЕЛИ WZNW В ВИДЕ ПАРЫ ЛАКСА

Одномерная модель WZNW, полученная редукцией автодуальных уравнений Янга-Миллса, сведена к нормальному представлению пары Лакса.

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