EXACT SOLUTION OF WZNW MODEL

M.A. MUKHTAROV
Institute of Mathematics and Mechanics
370602, Baku, F.Agaev str. 9, Azerbaijan

One dimensional reduction of WZNW is integrated in the case of $A_1(SL(2,C)$ algebra.

1. The problem of constructing of the solutions of integrable models and its dimensional reductions, the one dimensional WZNW model in our case, in the explicit form remains important for the present time. The interest arises from the fact that almost all integrable models in one, two and (1+2)-dimensions are symmetry reductions of SDYM or they can be obtained from it by imposing the constraints on Yang-Mills potentials [1-13].

This work is a direct continuation of [14-16], where the exact solutions have been derived by discrete symmetry transformation method that allows generating new solutions from the old ones in much more easier way than applying methods from [11]. The Lax pair presentation of the model under consideration is the first step in this program [16] that we hope will give us a key to construct solutions for an arbitrary semisimple algebra.

2. The one dimensional reduction of self duality equations obtained in [11] is the equation for the element f, taking values in the semisimple algebra,

$$\frac{\partial^2 f}{\partial r^2}+\frac{\partial f}{\partial r}-[H,[H,f]]-2[X^-,[X^+,f]]-2[X^+,[X^-,f]]+ \\ +2[[\frac{\partial}{\partial r}-H,f],[X^+,f]]=0 \tag{1}$$

Here $H, X^{\pm}$ are generators of $A_1(SL(2,C)$ algebra

$$\left[X^+,X^-\right]=H,\left[H,X^{\pm}\right]=\pm 2X^{\pm} \tag{2}$$

embedded to gauge algebra in the half-integer way.

The equation (1) has been reduced [16] to the following form:

$$\frac{\partial}{\partial\tau}\left(\frac{\partial q}{\partial\tau}q^{-1}\right)=[qF_0q^{-1},X^+] \tag{3}$$

Equation (2) is one-dimensional WZNW (Wess-Zumino-Novikov-Witten) equation [17-19].

We'll deal with the presentation of the equation under consideration in the form (1) and in the simplest case of f taking values in the algebra $A_1(SL(2,C)$:

$$f=xX^-+yH+zX^+, \tag{4}$$

where generators $H, X^{\pm}$ satisfy the same commutational relations (2).

Then the equation (1) can be rewritten for the components of (4) as the system of three nonlinear one dimensional second order equations, the general solution of which has to be dependent on six constants.

The system of equations has the form:

$$\begin{cases} x''+x'-2xx'-2x^2-2x=0 \\ y''+y'-2yx'-2yx-2y=0 \\ z''+z'+2z'x-2yz-2z+4yy'=0 \end{cases} \tag{5}$$

Consider the first equation of the system (5) representing it in a form:

$$(x'+2x+1)'-2x(x'+2x+1)=0 \quad \text{or} \quad \frac{\partial}{\partial\tau}\ln(x'+2x+1)=2x$$

Introducing new unknown function $u=\ln(x'+2x+1)$ we have:

$$u'=2x;\; u''+u'=2(x'+2x+1)-2=2e^u-2$$

Making again the change of variables: $u_1=u+2\tau$ we simplify last equation:

$$u_1''+u_1'=2e^{u_1-2\tau}$$

and after the substitution $t=e^{-\tau}$ we come to one-dimensional Liuville equation

$$\ddot{u}_1=2e^{u_1}$$

Multiplying both sides of the last equation by $\dot{u}_1$

$$\dot{u}_1\,\ddot{u}_1 = 2\dot{u}_1 e^{u_1}\,; \quad \frac{d}{dt}(\dot{u}_1)^2 = 4\frac{d}{dt}e^{u_1}$$

and integrating once, we have

$$\dot{u}_1 = 2\sqrt{e^{u_1} + k^2}\,,$$

where k is a constant.

The second integration gives the following:

$$t - c = \int \frac{du_1}{2\sqrt{e^{u_1} + k^2}} = -\frac{1}{k}\int \frac{dw}{\sqrt{1+w^2}} = -\frac{1}{k}\left(w + \sqrt{1+w^2}\right),$$

where $w = ke^{-\frac{u_1}{2}}$.

Using all above introduced notation we have:

$$e^{u_1} = \frac{k^2}{sh^2(kt + c)}$$

and using the relations $t = e^{-\tau} = e^{-\ln R} = R^{-1}$ and $e^{u_1} = e^{u+2\tau} = R^2 e^u$, we eventually have the relation:

$$e^u = \frac{k^2}{R^2 sh^2\left(kR^{-1} + c\right)}$$

Taking into account the relation

$$x = \frac{1}{2}u' = \frac{1}{2}R\frac{du}{dR}$$

we derive the solution of the first equation of the system under consideration:

$$x = -1 + cth\left(kR^{-1} + c\right) \tag{6}$$

Let's rewrite the second equation of the system (5) in the form:

$$y'' + y' = 2(x' + 2x + 1) = 2ye^u$$

and introducing the same, as in first equation, variables (t, u_1), we have:

$$\ddot{y} = 2y\frac{k^2}{R^2 sh^2\left(kR^{-1} + c\right)} \quad \text{or}$$

$$\frac{d^2 y}{dr^2} = 2y\frac{1}{sh^2 r}\,, \tag{7}$$

where $r = kR^{-1} + c$.

From the general theory of linear equations it follows that the Wronskian of two solutions of the equation (7) is a constant, that is if y_1 and y_2 are solutions of (7) then

$$W(y_1, y_2) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = c_1 \tag{8}$$

If one knows one partial solution of (7), saying y_1, then second solution can be obtained via the following relation:

$$y_2 = c_1 y_1 \int (y_1)^{-2} dr + c_2 y_1 \tag{9}$$

The equation (7) has the solution $y_1 = cthr$. Substituting it to (9) and making the corresponding integration, we come to the general solution of equation (7):

$$y_1 = c_1 cthr + c_2(rcthr - 1)\,,$$

or in terms of original variables:

$$y_1 = c_1 cth\left(kR^{-1} + c\right) + c_2\left(kR^{-1} cth\left(kR^{-1} + c\right) - 1\right) \tag{10}$$

Consider the homogeneous part of the first equation of the system (5):

$$z'' + z' + 2z'x - 2yz - 2z = 0$$

Rewriting it in a form

$$(z' - z)' + 2(z' - z)(x + 1) = 0$$

or

$$\frac{d}{dR}\left(R\frac{d}{dR}z - z\right) = -\frac{2}{R}(x+1) = -\frac{2k}{R^2}\,cth\left(kR^{-1} + c\right).$$

Here we used the expression for x from (6).

The first integration gives the equation:

$$R\frac{d}{dR}z - z = c_3 sh^2\left(kR^{-1} + c\right),$$

the second one gives the required solution:

$$z = \frac{c_3}{2k}\left(1 - \frac{R}{2}sh2\left(kR^{-1} + c\right)\right) + c_4 R$$

and the solution of the whole equation, whose inhomogeneous part is defined by the known solution y of the second equation of the system, is given by the following expression:

$$z=\frac{c_3}{2k}\left(1-\frac{R}{2}sh2c\right)+c_4R+$$
$$+\frac{c_1^2}{2k}Rth\left(kR^{-1}+c\right)+2c_1c_2th\left(kR^{-1}+c\right)+c_2^2(kR^{-1}th\left(kR^{-1}+c\right)-1)$$

As the solution of the system of three second order ordinary differential equations depends on six arbitrary constants, this solution is the general one.

Comparison of the solutions obtained with that ones obtained by means of Riemann-Hilbert problem is the subject of further publications.

[1] *R.S. Ward, Phil. Trans.* R. Soc. Lond.A315, 451 (1985); Lect. Notes Phys., 1987, 280, 106; Lond. Math. Soc. Lect. Notes Ser., 1990, 156, 246.

[2] *L.J. Mason and G.A. J.Sparling.* Phys. Lett., 1989, A137, 29; J. Geom. and Phys., 1992, 8, 243.

[3] *S. Chakravarty, M.J. Ablowitz and P.A. Clarkson.* Phys. Rev. Lett., 1990, 1085.

[4] *I. Bakas and D.A. Depireux.* Mod. Phys. Lett., 1991, A6, 399.

[5] *M.J. Ablowitz, S. Chakravarty and L.A. Takhtajan.* Comm. Math. Phys., 1993, 158, 1289.

[6] *T.A. Ivanova and A.D. Popov.* Phys. Lett., 1992, A170, 293.

[7] *L.J. Mason and N.M.J. Woodhouse.* Nonlinearity 1, 1988, 73; 1993, 6, 569.

[8] *M. Kovalyov, M. Legare and L. Gagnon.* J. Math. Phys., 1993, 34, 3425.

[9] *M. Legare and A.D. Popov.* Pis'ma Zh. Eksp. Teor. Fiz., 1994, 59, 845.

[10] *A.A. Belavin and V.E. Zakharov.* Phys. Lett., 1978, B73, 53.

[11] *A.N. Leznov and M.A. Mukhtarov.* J. Math. Phys., 1987, 28 (11), 2574; Prepr. IHEP, 1987, 87-90. Prepr. ICTP 163, Trieste, Italy, 1990; J. Sov. Lazer Research, 13 (4), 284, 1992.

[12] *A.N. Leznov.* IHEP preprint-92/87, 1990.

[13] *A.N. Leznov, M.A.Mukhtarov and W.J.Zakrzewski.* Tr. J. of Physics 1995, 19, 416.

[14] *M.A.Mukhtarov.* Fizika, 2002, v. 5, N 2, 38

[15] *M.A.Mukhtarov.* Fizika, 2002, v. 5, N 3, 3

[16] *M.A.Mukhtarov.* Fizika, 2005, v. 8, N 2, 40

[17] *V.G. Knizhnik and A.B. Zamolodchikov*, Nucl. Phys. B247 (1984) 83.

[18] *F. Bastianelli*, Nucl. Phys. B361 (1991) 555.

[19] *F. Bastianelli*, Nucl. Phys. B361 (1991) 555.

[20] *A.A. Tseytlin*, Nucl. Phys. B411 (1994) 509.

M.A. Muxtarov

WZNW MODELİNİN DƏQİQ HƏLLƏRİ

SL(2,C) cəbri halında WZNW modelinin birölçülü reduksiyası inteqrallanmışdır.

М.А. Мухтаров

ТОЧНЫЕ РЕШЕНИЯ МОДЕЛИ WZNW

Одномерная редукция модели WZNW проинтегрирована в случае алгебры SL(2,C)

Received: 18.09.06