GENERALIZED TODA MODELS

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The group theoretical approach has been developed for one-dimensional generalized non-abelian conformal affine Toda models.

1. The study of classical and quantum non-linear integrable models is of great interest in Mathematics and especially in High Energy Physics, where such models have been used as laboratories to develop methods to explore the non-linear perturbative aspects of gauge theories, gravity and string theory. In particular, they could help in understanding some stable classical solutions, like monopoles, which must have an important role in the quantum theory, and which cannot be understood by the existing methods.

Within the integrable models in 1+1 dimensions, the investigation of the different Toda Field Theories has recently received a lot of attention. According to their underlying algebraic structure, they can be divided into three categories; each one exhibiting nice characteristic properties. First, associated to the finite simple Lie algebras, there are the Conformal Toda models, which are conformally invariant 1+1 field theories. Even more, they permit the construction of extensions of the Virasoro algebra including higher spin generators, namely W-algebras. The second class of theories are the Affine Toda models, based on loop algebras, which can be regarded as a perturbed Conformal Toda model where the conformal symmetry is broken by the perturbation while the integrability is preserved [1]. One of their main properties is that they possess soliton solutions. These two classes of models are called abelian or non-abelian referring to whether their fields live on an abelian or non-abelian group [2, 3, 4, 5].Finally, the conformal symmetry can be restored in the abelian Affine Toda models just by adding two extra fields which do not modify the dynamics of the original model; one of these fields is a connection whose only role is to implement the conformal invariance. These are the so called Conformal Affine Toda models [6, 7], and they are based on a full Kac-Moody algebra; moreover, they are integrable [8], and have soliton solutions [9]. In fact, many properties of the Affine Toda models can be more easily understood by considering them as the Conformal Affine Toda models with the conformal symmetry spontaneously broken.

At the same time the problem of constructing of the solutions of self-dual Yang-Mills (SDYM) model and its dimensional reductions, the one dimensional WZNW model in our case, in the explicit form for arbitrary semisimple Lie algebra, rank of which is greater than two, remains important for the present time. The interest arises from the fact that almost all integrable models in one, two and (1+2)-dimensions are symmetry reductions of SDYM or they can be obtained from it by imposing the constraints on Yang-Mills potentials [10-27].

Two effective methods of generating of the exact solutions, the Riemann Hilbert Problem formalism [20] and the discrete symmetry transformation method [22], have been applied to Toda like systems. This work is devoted to construct a group theoretical background of earlier considerations.

2. The two-loop WZNW model was introduced in [6] as the generalization of the ordinary WZNW model to the affine case. Its equations of motion are given by

$$\partial_+\left(\partial_-\hat{g}\hat{g}^{-1}\right)=0\ ;\quad \partial_-\left(\partial_+\hat{g}\hat{g}^{-1}\right)=0\ , \qquad (2.1)$$

where $\partial_\pm$ are derivatives with respect to the light-cone variables $x_\pm = x \pm t$, and $\hat{g}$ is an element of the group G formed by exponentiating an untwisted affine (real) Kac-Moody (KM) algebra $\hat{G}$. Its generators T_a^m, D and C satisfy the commutation relations

$$\left[T_a^m\,,T_b^n\right]= f_{ab}^{\,c}\,T_c^{m+n}+mCg_{ab}\delta_{m+n,0} \qquad (2.2)$$

$$\left[D\ ,T_a^m\right]=mT_a^m\ ,\qquad \left[C,D\right]=\left[C\ ,T_a^m\right]=0 \qquad (2.3)$$

where f_{ab}^c are the structure constants of a finite (real) semisimple Lie algebra G, n and m are integers, and g_{ab} is the Killing form of G, i.e., $g_{ab}=Tr(T_aT_b)$, T_a being the generators of G. The non-degenerate bilinear form of $\hat{G}$ is defined as

$$Tr(T_a^mT_b^n)=\delta_{m+n,0}Tr(T_a,T_b)\,,\quad Tr(C,D)=1$$
$$Tr(C,T_a^m)=Tr(D,T_a^m)=0 \qquad (2.4)$$

and we will use the same notation, Tr, for both the Killing form of G and the bilinear form of $\hat{G}$.

The two-loop WZNW model is invariant under left and right translations

$$\hat{g}(x_+,x_-)\to\hat{g}_L(x_-)\hat{g}(x_+,x_-)\ ,\ \ \hat{g}(x_+,x_-)\to\hat{g}(x_+,x_-)\hat{g}_R(x_+) \qquad (2.5)$$

The corresponding Noether currents are the components of $\partial_-\hat{g}\hat{g}^{-1}$ and $\hat{g}^{-1}\partial_+\hat{g}$, and they generate two commuting copies of the so called two-loop Kac-Moody algebra, defined by the relations

$$\left[J_a^m(x),\ J_b^n(y)\right]=f_{ab}^{\ c}\ J_c^{m+n}(x)\partial(x-y)+g_{ab}\delta_{m,-n}\left(k\partial_x\partial(x-y)+mJ^C(x)\partial(x-y)\right) \tag{2.6}$$

$$\left[J^D(x),\ J_a^m(y)\right]=mJ_a^m(y)\partial(x-y) \tag{2.7}$$

$$\left[J^C(x),\ J^D(y)\right]=k\partial_x\partial(x-y) \tag{2.8}$$

$$\left[J^C(x),\ J_a^m(y)\right]=0 \tag{2.9}$$

The left and right currents satisfying the above relations are related to the group element $\hat{g}$ in eq.(2.1) by

$$F_R(x_+)=k\hat{g}^{-1}\partial_+\hat{g}=\sum_{ab}\ \sum_{n=-\infty}^{\infty}g^{ab}J_{R,a}^{-n}(x_+)T_b^n+\ J_R^D(x_+)C+J_D^C(x_+)D \tag{2.10}$$

$$F_L(x_-)=-k\partial_-\hat{g}\hat{g}^{-1}=\sum_{ab}\ \sum_{n=-\infty}^{\infty}g^{ab}J_{L,a}^{-n}(x_-)T_b^n+\ J_L^D(x_-)C+J_D^C(x_-)D \tag{2.11}$$

where g^{al} is the inverse of the Killing form g_{ab} defined above. The different meaning of the two central extensions in eqs.(2.6)-(2.9) algebra is clarified by expressing the algebra as

$$\left[Tr(UF(x)\ ,\ Tr(V\ F(y)\right]=Tr([U,V\ ]\ F(x))\partial(x-y)+kTr(UV\ )\partial_x\partial(x-y) \tag{2.12}$$

where U,V are two elements of the Kac-Moody algebra $\hat{G},F$ is either F_R or F_L, and Tr is the invariant bilinear form of $\hat{G}$.

Consider now a gradation of the Kac-Moody algebra $\hat{G}$

$$\hat{G}\ =\oplus_s\hat{G}_s \tag{2.13}$$

with

$$\left[\hat{G}_s\ ,\hat{G}_r\right]\subset\hat{G}_{s+r} \tag{2.14}$$

The reduction presented in this section does not require that this gradation is integer; it just needs that the grades s take zero, positive and negative values, i.e.,

$$\hat{G}=\hat{G}_+\oplus\hat{G}_0\oplus\hat{G}_- \tag{2.15}$$

with

$$\hat{G}_+=\underset{S>0}{\oplus}\hat{G}_S\ \ ,\quad \hat{G}_-=\underset{S<0}{\oplus}\hat{G}_S \tag{2.16}$$

We now consider those group elements that can be written in a "Gauss decomposition" form

$$\hat{g}=NBM\in G \tag{2.17}$$

where N,B and M are group elements formed by exponentiating elements of $\hat{G}_+$, $\hat{G}_0$ and $\hat{G}_-$ respectively.

Using eq.(2.17), we can write the equations of motion (2.1) as

$$\partial_-K_R=\left[K_R\ ,\partial_-MM^{-1}\right] \tag{2.18}$$

$$\partial_+K_L=\left[K_L\ ,N^{-1}\partial_+N\right] \tag{2.19}$$

where we have introduced

$$K_L=N^{-1}\partial_-\hat{g}\hat{g}^{-1}N=N^{-1}\partial_-N+\partial_-BB^{-1}+B\partial_-MM^{-1}B^{-1} \tag{2.20}$$

$$K_R=M\ \hat{g}^{-1}\partial_+\hat{g}M^{-1}=B^{-1}\ N^{-1}\partial_+NB+B^{-1}\partial_+B+\partial_+MM^{-1} \tag{2.21}$$

Although the quantities $K_{L/R}$ are not chiral, they have a simpler structure than the currents and will be very useful in what follows. We will reduce the two-loop WZNW model by imposing constraints not directly on the currents but on $K_{L/R}$. We impose the constraints

$$B^{-1}\left(N^{-1}\partial_+N\right)B=\Lambda_l \tag{2.22}$$

$$B\left(\partial_-M\right)M^{-1}B^{-1}=\Lambda_{-l} \tag{2.23}$$

where $\Lambda_{\pm l}$ are constant elements of $\hat{G}_{\pm l}$. These constraints reduce the two-loop WZNW model to a theory containing only the fields corresponding to the components of B and to the components of N and M associated to the generators whose grade is $<l$ and $>l$ respectively.

To obtain the equations of motion for such model one notices that the constraints (2.22) and (2.23) imply that

$$N^{-1}\partial_+N\in\hat{G}_l \tag{2.24}$$

$$\left(\partial_-M\right)M^{-1}\in\hat{G}_{-l} \tag{2.25}$$

Therefore the only terms of zero grade on the right hand side of (2.19) are coming from $\left[\Lambda_{-l}\ ,N^{-1}\partial_+N\right]=\left[\Lambda_{-l}\ ,B\Lambda_lB^{-1}\right]$. So we get

$$\partial_+(\partial_- BB^{-1}) = \left[\Lambda_{-l}, B\Lambda_l B^{-1}\right] \qquad (2.26)$$

which can also be written as

$$\partial_-(B^{-1}\partial_+ B) = -\left[\Lambda_l, B^{-1}\Lambda_{-l} B\right] \qquad (2.27)$$

These are the equations of motion of what we call the generalized non-abelian conformal affine Toda models.

3. The one dimensional reduction of self duality equations obtained in [20] are the equations for the element f, taking values in the semisimple algebra,

$$\begin{aligned}&\frac{\partial^2 f}{\partial r^2} + 2\frac{\partial f}{\partial r} - [H,[H,f]] - 2[X^-,[X^+,f]] - 2[X^+,[X^-,f]] + \\ &+ 2[[\frac{\partial}{\partial r} - H, f],[X^+,f]] = 0\end{aligned} \qquad (3.1)$$

Here $H, X^{\pm}$ are generators of $A_1(SL(2,C)$ algebra

$$\left[X^+_M, X^-\right] = H, \left[H, X^{\pm}\right] = \pm 2X^{\pm}$$

embedded to gauge algebra in the half-integer way.

Let's rewrite (3.1) in the equivalent form:

$$\begin{aligned}&[\frac{1}{2}(\frac{\partial}{\partial r} + H) - [X^+, f], -\frac{1}{2}[\frac{\partial}{\partial r} - H, f] + X^-] - \\ &-\frac{1}{2}[\frac{\partial}{\partial r} - H, f] + X^- = 0\end{aligned}$$

This equation after changing the variable $t = \ln r$ has the following form

$$\begin{aligned}&[\frac{\partial}{\partial t} + \frac{1}{2}H - [X^+, f], -\frac{\partial f}{\partial t} + \frac{1}{2}[H, f] + X^-] - \\ &-\frac{\partial f}{\partial t} + \frac{1}{2}[H, f] + X^- = 0\end{aligned} \qquad (3.2)$$

Introducing the notation

$$\widetilde{F} = e^{\frac{1}{2}Ht}(-\frac{\partial f}{\partial t} + \frac{1}{2}[H, f] + X^-)e^{-\frac{1}{2}Ht}, \qquad (3.3)$$

multiplying (2) from the left side by $e^{\frac{1}{2}Ht}$ and from the right side by $e^{-\frac{1}{2}Ht}$, we obtain

$$\frac{\partial \widetilde{F}}{\partial t} - [[e^{\frac{1}{2}Ht} X^+ e^{-\frac{1}{2}Ht}, e^{\frac{1}{2}Ht} f e^{-\frac{1}{2}Ht}], \widetilde{F}] + \widetilde{F} = 0$$

Due to the evident equality

$$e^{\frac{1}{2}Ht} X^+ e^{-\frac{1}{2}Ht} = e^t X^+$$

the last equation can be rewritten in a form

$$\frac{\partial \widetilde{F}}{\partial t} - e^t[[X^+, \widetilde{f}], \widetilde{F}] + \widetilde{F} = 0, \qquad (3.4)$$

where $\widetilde{f} = e^{\frac{1}{2}Ht} f e^{-\frac{1}{2}Ht}$.

In terms of these notations we have from (3.3) the following expression

$$\widetilde{F} = -\frac{\partial \widetilde{f}}{\partial t} + [H, \widetilde{f}] + X^- e^{-t} = 0$$

Let's introduce the notation

$$F = e^t \widetilde{F} = -e^t \frac{\partial \widetilde{f}}{\partial t} + e^t[H, \widetilde{f}] + X^- = 0$$

Then (3.4) has a form

$$\frac{\partial F}{\partial t} + [A, F] = 0, \qquad (3.5)$$

where $A = -e^t[X^+, \widetilde{f}]$.

The equation (5) is one-dimensional evolution equation defined by Lax pair operators and it is one of the principal criteria of equations integrability.

From the presentation (3.5) it is followed that

$$\frac{\partial}{\partial t} spF^n = 0, \ for \ \forall n$$

and solution of the equations can be found in a form

$$F = \varphi F_0 \varphi^{-1}, \qquad (3.6)$$

where $\varphi(t)$ takes values in the corresponding Lie group and $F_0 = F|_{t=0}$.

From equation (5) and presentation (6) it is directly followed the expression for the operator A:

$$A = \varphi' \varphi^{-1} \qquad (\varphi' = \frac{\partial \varphi}{\partial t}) \qquad (3.7)$$

Let's consider the commutator of F with X^+:

$$[X^+,F]=[X^+,X^-]-e^t\frac{\partial}{\partial t}[X^+,\widetilde{f}]+e^t[X^+,[H,\widetilde{f}]]=$$

$$=H-e^t\frac{\partial}{\partial t}[X^+,\widetilde{f}]-2e^t[X^+,\widetilde{f}]+e^t[X^+,[H,\widetilde{f}]]=$$

$$=H-\frac{\partial}{\partial t}(e^t[X^+,\widetilde{f}])-e^t[X^+,\widetilde{f}]+[H,e^t[X^+,\widetilde{f}]].$$

Taking into account (3.6) and (3.7) the last expression can rewritten in a form

$$[X^+,\varphi F_0\varphi^{-1}]=H-(\varphi^/\varphi^{-1})^/-\varphi^/\varphi^{-1}+[H,\varphi^/\varphi^{-1}].$$

Making the substitution $\varphi=e^{Ht}q$ and introducing a new variable $\tau=e^{-t}$, we have

$$\frac{\partial}{\partial\tau}(\frac{\partial q}{\partial\tau}q^{-1})=[qF_0q^{-1},X^+] \qquad (3.8)$$

Equation (3.8) is one-dimensional generalized non-abelian conformal affine Toda model as it is obviously seen from eq. (2.26). The group-theoretical approach derived for this equation in paragraph 2 gives reasonable opportunities to obtain the exact solutions for arbitrary semisimple algebra and that will be the subject of the further publications.

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TODANIN ÜMUMİLƏŞDİRİLMİŞ MODELİ

Todanın birölçülü qeyri-abel konform affin modeli üçün qruplaşmış nəzəri yanaşma işlənmişdir.

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ОБОБЩЕННЫЕ МОДЕЛИ ТОДА

Теоретико групповой подход разработан для одномерных неабелевых конформных аффинных моделей Тода.

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