EXACT SOLVABLE NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS AND THEIR NEW SOLUTIONS

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We consider three types of nonlinear partial differential equation of polynomial form. Using the method of motion integrals the exact solutions of its equations are found.

Introduction.

There are many methods of obtaining exact solutions of nonlinear equations: see [1-3]. The nonlinear forms of some linear partial differential equation are considered in work [4]. We were found explicit substitutions of dependent variables which transform the equation under study to linear equations. In this paper which is continuation of paper [4] we investigate five types of two variable nonlinear partial differential equations of third order which reduce to linear equations by substitutions of dependent variables. We obtain the exact solutions of one of types of presented equations. We confine ourselves to equations of polynomial type.

Classes of nonlinear equations.

Let us consider a general linear partial equation of the second order:

$$a\psi_t + b\psi_{xt} + c\psi_{tt} + d\psi_{xx} = 0 \qquad (1)$$

Making the substitutions of the following form:

$$\psi = exp(\gamma\varphi + \delta\varphi_x + \beta\varphi_t) \qquad (2)$$

(other substitutions lead to non-polynomial equation) we obtain five different partial differential equations which are linear with respect to the derivatives of the third order and nonlinear with respect to of the second and first orders in the dependence of relations between coefficients a, b, c, d, γ, δ, β:

I. $\alpha(\gamma\varphi_t + \delta\varphi_{xt}) + d(\gamma^2\varphi_{xx}^2 + 2\gamma\delta\varphi_{xx}\varphi_x + \gamma\varphi_{xx} + \delta\varphi_{xxx}) = 0$
$b = c = \beta = 0.$

II. $\alpha(\gamma\varphi_t + \delta\varphi_{xt}) + b(\gamma^2\varphi_t\varphi_x + \gamma\delta\varphi_{xt} + \varphi_x + \gamma\delta\varphi_x\varphi_{xx} + \delta^2\varphi_{xt}\varphi_{xx} + \gamma\varphi_{xt} + \delta\varphi_{xxx}) = 0$
$c = d = \beta = 0.$

III $\alpha(\gamma\varphi_e + \delta_{че}) + c[\gamma^2\varphi_t^2 + \delta^2\varphi_{xt}^2 + \gamma\varphi_{tt} + 2\gamma\delta\varphi_{xt}\varphi_t + \gamma^2\varphi_x^2 + \delta^2\varphi_{xx}^2 + \gamma\varphi_{xx} + 2\gamma\delta\varphi_{xx}\varphi_x$
$+ \delta(\varphi_{xxx} + \varphi_{xtt})] = 0$
$c = d; \beta = b = 0.$

IV. $\alpha(\gamma\varphi_t + \beta\varphi_{tt}) + d(\gamma^2\varphi_x^2 + \beta_{tx}^2 + 2\gamma\beta\varphi_{tx}\varphi_x + \gamma\varphi_{xx} + \beta\psi_{txx}) = 0.$
$b = c = \delta = 0.$

V. $\alpha[\gamma\varphi_t + \beta(i\varphi_{xt} + \varphi_{tt})] + d[\gamma^2(\varphi_t\varphi_x + \varphi_x^2) + 2\gamma(i\varphi_{tt}\varphi_x - \varphi_t\varphi_{xx} + i\varphi_t\varphi_{xt} + 2i\varphi_{xx}\varphi_x + 2\varphi_{tx}\varphi_x)$
$+ \gamma^2(-i\varphi_{xt} - \varphi_{tt}\varphi_{xx} + i\varphi_{tt} - \varphi_{xt} - \varphi_{xx}^2 + \varphi_{tx}^2 + 2i\varphi_{xx}\varphi_{tx}) + \alpha(i\varphi_{xt} + \varphi_{xx}) + \gamma(i\varphi_{xtt} - \varphi_{xxt} + i\varphi_{xxx} + \varphi_{txx})] = 0.$
$c - 0; b = id; \beta = i\gamma.$

Other equations of this class are reduced to shown equations by substitutions of independent variables (particularly x↔t).

Let us obtain the exact solution of equation II. This equation is reduced to following linear equation:

$$\alpha\psi_t + d\psi_{xx} = 0 \qquad (3)$$

by the replacement

$$\psi = \exp(\gamma\varphi + \beta\psi_t) \qquad (4)$$

Among solutions of equation (3) the coherent-state functions exists (we propose a, d=const):

$$\psi_\alpha = (\pi)^{-\frac{1}{4}} exp\left\{ -|\alpha|^2/2 - \alpha^2(2dt/2a+1) + \alpha x(2)^{-\frac{1}{2}} \right\} x(dt/2a+1)^{-\frac{1}{2}}$$
$$exp\left\{ -[x-\alpha(2)^{-\frac{1}{2}}(dt/2a-1)]^2/2(dt/2a+1) \right\} \tag{5}$$

Substituting (5) in (3) we obtain the exact solution of equation II:

$$\begin{aligned}\psi_\alpha(x_1 t) = \gamma^{-1}[ln(\pi^{\frac{1}{4}}) - |\alpha|^2/2 - \alpha^2(dt/2a+1-\beta\gamma^{-1})/2 - \\ - ln(dt/2a+1)/2] - (\beta d)^{-1} a^{-1}\left[(-\beta d/2a\gamma + x^2) - 4dx/\sqrt{2} + \alpha^2 exp(-\gamma t/\beta + \right. \\ \left. + d\gamma/2a\beta)\mathrm{E}_i(2a\gamma(dt/2a+1)/d\beta)\right]\end{aligned} \tag{6}$$

where $\mathrm{E}_i(ax) = \int [exp(ax)/x] dx$.

Knowing that ψ_α is generating function of Hermitian polynomials one can obtain others solutions of following form:

$$\begin{aligned}\psi_n(x_1 t) = \gamma^{-1}[ln(\pi^{\frac{1}{4}}) - |\alpha|^2/2 - ln(dt/2a+1)/2)] - \\ - 2a\beta^{-1}(-\beta/\gamma + x^2) exp(d\gamma/2a\beta - \gamma t/\beta)\mathrm{E}_i(2a\gamma(dt/2a+1)/d\beta) + . \\ + exp(-\gamma t/\beta)\int ln[\mathrm{H}n(\sqrt{2}(-d^2t^2/4a^2 - 1)/4x)] exp(\gamma t/\beta) dt\end{aligned} \tag{7}$$

If the integrals of motion of equation (3), i.e. operators $\hat{I}_\alpha$ satisfying the following relation:

$$\left[\hat{I}_\alpha, a\delta/dt + d\delta^2/\delta x^2\right]\psi = 0 \ . \tag{8}$$

then from arbitrary solution $\varphi_{(x,t)}$ of equation (3) one can obtain new solutions by following rule [5]:

$$\widetilde{\psi} = f(\hat{\mathrm{I}}_1, \hat{\mathrm{I}}_2 ... \hat{\mathrm{I}}_n)\psi \tag{9}$$

where *f* is arbitrary function.

Therefore if φ is solution of equation II then the following arbitrary functions are solutions of this equation:

$$\widetilde{\varphi} = \hat{L}\widetilde{\psi} = \hat{L}f(\hat{I}_1, \hat{I}_2, ... I_n)L^{-1}\varphi \tag{10}$$

where $\hat{L}$ is transformation operator from ψ to φ .

Integrals of motion for equation (3) are:

$$I_1 = \partial/\partial t; I_2 = \partial/\partial x; I_3 = x - 2tda^{-1}\partial/\partial x \tag{11}$$

One can obtain new exact solutions of equation II from known solutions (for example (5) and (6)):

$$\widetilde{\varphi} = (\gamma + \beta\delta/\delta t)^{-1} \ln f(\hat{\mathrm{I}}_1, \hat{\mathrm{I}}_2, \hat{\mathrm{I}}_3) \exp(\gamma\varphi + \beta\frac{\delta\varphi}{\delta t}) \tag{12}$$

The solutions of rest equations I – V can be obtained by analogical way.

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XÜSUSİ TÖRƏMƏLƏRDƏ DƏQİQ HƏLL OLUNAN QEYRİ-XƏTTİ DİFERENSİAL TƏNLİKLƏR VƏ ONLARIN YENİ HƏLLƏRİ

Xüsusi törəmələrdə polinomial şəkilli qeyri-xətti diferensial tənliklərin üç tipi nəzərdən keçirilmişdir. Hərəkət inteqralları metodundan istifadə edərək, bu tənliklərin dəqiq həlləri tapılmışdır.

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ТОЧНО РЕШАЕМЫЕ НЕЛИНЕЙНЫЕ ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ В ЧАСТНЫХ ПРОИЗВОДНЫХ И ИХ НОВЫЕ РЕШЕНИЯ

Рассмотрены три типа нелинейных дифференциальных уравнений в частных производных полиномиального вида. Используя метод интегралов движения, найдены точные решения этих уравнений.

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