EIGENVALUE SOLUTIONS FROM A JOSEPHSON JUNCTION CIRCUIT AS A MODEL OF A CHARGE QUBIT

MEHMET CANTÜRK[1], EROL KURT[2]

[1]*Department of Computer Engineering, Atilim University, 06836, Ankara, Turkey*

[2]*Department of Electrical Education, Faculty of Technical Education, Gazi University, 06500, Ankara, Turkey*

Characterization of a single Josephson junction circuit numerically studied by means of periodic tridiagonal Hermitian eigenvalue problem that arises from quantum mechanical formulation. The wave function, ψ(θ), representing the electron pairs inside the junction, is determined by solving the so-called general form of Mathieu-type differential equation. Eigenenergy spectra and corresponding eigenfunctions are found in the regime of charge qubit and examined by using different gate voltages (i.e. n_gs). Some representative results are obtained for lowest eigenenergies and corresponding eigenfunctions.

1. INTRODUCTION

There has been an increasing interest in the possibility of fabricating solid-state devies for quantum computing for ten years [1]. Superconducting devies such as superconducting quantum interference devices (SQUIDs), quantum boxes and superconducting single electron transistors (SSETs) are suitable candidates for this purpose. The physical realizations of a solid-state qubit is provided by a Cooper pair box which is a small superconducting island connected to a large superconducting electrode, a reservoir, through a Josephson junction. Two charge states of the box, differing by one Cooper pair, are coherently mixed by the Josephson coupling as was confirmed experimentally.

On the other hand, the Rapid Single-Flux-Quantum (RSFQ) electronics are considered as possible complementary electronics to Josephson quantum computing networks for the control and readout of the quantum state [2,3], as reflected in forecast studies (e.g., in the European Network for Superconductivity (SCENET) and Quantum Information Science and Technology (QIST) roadmaps [4]). The fabrication and controlled manipulation of Josephson junction qubits is possible by using present-day technologies [1]. It should be stated that theoretical and numerical approaches can estimate the appropriate system parameters for laboratory researches [5–8]. Whatever the application, superconductivity has been linked to the concepts of long range order, coherence and macroscopic behavior [9]. By evaluating the findings from the theoretical studies, the minimization of negative impacts such as decoherence and nonlinearity can be achieved at a certain range. For instance, those efforts have made possible to create a plenty of applications for a variety of purposes such as chaotic communication system, voltage standards, SQUID, ultra-sensitive flux measurement systems and synchronized superconducting devices etc. [5,6,8,10]. It is widely known that quantum dynamics of a superconducting device can be described in two completely different but equivalent mathematical formalisms such as Heisenberg and Schrodinger formalisms [11]. There are many studies in the literature that their formulations are based on Heisenberg picture [1,12–14]. In one of our previous study [15], a single Cooper pair box (SCB) was modeled both analytically and numerically. Based on Schrodinger formalism, the solution was obtained with respect to phase difference. We proved that this model gave some interesting results so as to determine voltage-current ($V - I$) behavior for various *ng*s. For example, it was shown that Josephson and combined quasi-particle tunneling were encountered and some similarities with experimental findings were discussed.

In the present study, we will improve the preceding study for the integral and half-integral dimensionless charge numbers, *ng*s. In this manner, we have modelled the junction circuit and formulated the general form of Shrodinger equation in Section 2. In section 3, the numerical results are discussed in terms of eigenvalues. A relationship between eigenvalue and mean supercurrent is also recommended. Consequently, some concluding remarks are outlined in the last section.

2. THEORETICAL BACKGROUND

The simplest form of the junction circuit is depicted in Fig.1. Tunnel junction is characterized by a junction capacitance C_j and Josephson coupling energy E_j . A control gate voltage V_g is applied to the system via a gate capacitor C_g with an electrostatic energy, E_c. The superconducting condensate of the Cooper pairs in the superconducting island is represented by the electron pair wave function.

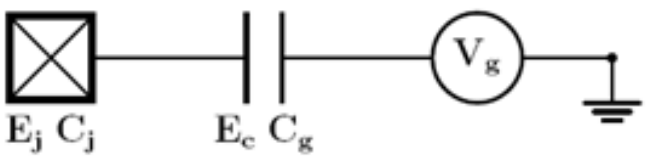

Figure 1. The simplest circuit model of a Josephson junction (JJ) including a gate voltage V_g and gate capasitance C_g. Corresponding gate charge is $Q_g = 2en_g = C_gV_g$.

Leaving the detail of the formulation to our earlier study [15], the Hamiltonian and time-independent part of Schrodinger equation for a well-defined wave function $\psi(\theta)$ read as,

$$H = -E_c\left[\frac{\partial^2}{\partial\theta^2} - i2n_g\frac{\partial}{\partial\theta} + \left\{2\frac{E_j}{E_c}\cos\theta - n_g^2\right\}\right] \quad 1(a)$$

$$\frac{d^2\psi}{d\theta^2} - i2n_g\frac{d\psi}{d\theta} + \left\{2\frac{E_j}{E_c}\cos\theta - n_g^2\right\}\psi = 0 \quad 1(b)$$

Eq.1(b) is, in principle, a general form of the Mathieu-type equation as the spatial (i.e. phase difference) and temporal statement of the junction in the interval [0, 2π] with an appropriate boundary condition. Furthermore, the problem can be simplified for numerical purposes as:

$$\frac{d^2\psi}{d\theta^2} - ip\frac{d\psi}{d\theta} + q\psi = -\varepsilon\psi \quad (2)$$

Here, $\varepsilon=2eV/E_c$ holds as the eigenvalue of the system, the real functions $p=2n_g$ and $q=2E_j/E_cCos\ (\theta) - n_g^2$ will construct the matrix elements under the periodic boundary conditions of $\psi(0) = \psi(2\pi)$ and $\psi'(0) = \psi'(2\pi)$.

Mathieu type eigenvalue problem in Eq.(2) can be solved efficiently by using finite difference representation on a grid of equispaced points. Based on central difference operators, $\delta\psi j = (\psi_{j+1} - \psi_{j-1})/2$ and $\delta^2\psi_j = (\psi_{j+1} - 2\psi_j + \psi_{j-1})$, Eq.(2) can be transformed into

$$e_j\psi_{j-1} + f_j\psi_j + e_j^*\psi_{j+1} + = \lambda\psi_j$$
$$\forall_j = 0, 1, 2, \ldots, N-1 \quad (3)$$

where h is the step size for discretization scheme (h should not be confused with the Planck constant), N is the number of subintervals in the grid, satisfying $N + 1 = 2\pi/h$ and e_j* is the complex conjugate of e_j. The terms in Eq.(3) are given as $e_j=1+ih\ p_j\ /2$, $f_j=h^2q_j\ -2$ and $\lambda=-h^2\varepsilon$. In the discritization scheme, the lower and upper ends of wave functions are specified by periodic boundary conditions: $\psi_{-1} = \psi_{N-1}$ and $\psi_N = \psi_0$. With the help of these expressions, one can write the matrix form of Eq. (3) as an eigenvalue problem $\mathbf{A}\psi_n = \lambda_n\psi_n$ over all indices. Here **A** is given by,

$$A = \begin{pmatrix} f_0 & e_0^* & 0 & 0 & 0 & \ldots & 0 & e_0 \\ e_1 & f_1 & e_1^* & 0 & 0 & \ldots & 0 & 0 \\ 0 & e_2 & f_2 & e_2^* & 0 & \ldots & 0 & 0 \\ 0 & 0 & e_3 & f_3 & e_3^* & \ldots & 0 & 0 \\ 0 & 0 & 0 & e_4 & f_4 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & .. & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots & f_{N-2} & e_{N-2}^* \\ e_{N-1}^* & 0 & 0 & 0 & 0 & \ldots & e_{N-1} & f_{N-1} \end{pmatrix} \quad (4)$$

as a complex NxN-dimensioned periodic tridiagonal coefficient matrix and λ_n defines the eigenvalue of n^{th} state.

It is widely known that periodic tridiagonal matrices typically arise from discretization of a second or higher order differential equations subjected to periodic boundary conditions [16–18]. For this reason, the matrix **A** is said to be periodic tridiagonal due to the nonzero terms at the corners such as $A_{0,N-1} = e_0 \neq 0$ and $A_{N-1,0} = e^*_{N-1} \neq 0$. These terms come into existence because of the periodic boundary conditions. The numerical solution of the system is evaluated by using the conventional *LAPACK* eigenvalue solver.

In this study, the number of subintervals is chosen to be N = 4000 and the physical parameters are considered as in [15]. The preference of such parameters is based on some experimental and technical studies [6,8]. According to these studies, the parameters namely Cooper pair charge energy, junction energy and energy gap are $E_c = 54\mu eV$, $E_j = 0.1253\mu eV$ and $\Delta = 0.189meV$, respectively. Note that, these parameters hold $2\Delta > E_c >> E_j$ as in [1,19] and characterize a charge qubit system. Therefore, the most of the simulations have been carried out for $E_j/E_c = 2.3\times10^{-3}$.

3. RESULTS AND DISCUSSIONS

For a complete picture of eigenenergies with corresponding eigenstates, the lowest four eigenstates are depicted as a function of n_g in Fig.(2). As also known from the literature, our situation is rather simple since the eigenstates are very close to the pair states at integral and half-integral n_gs [13,14].

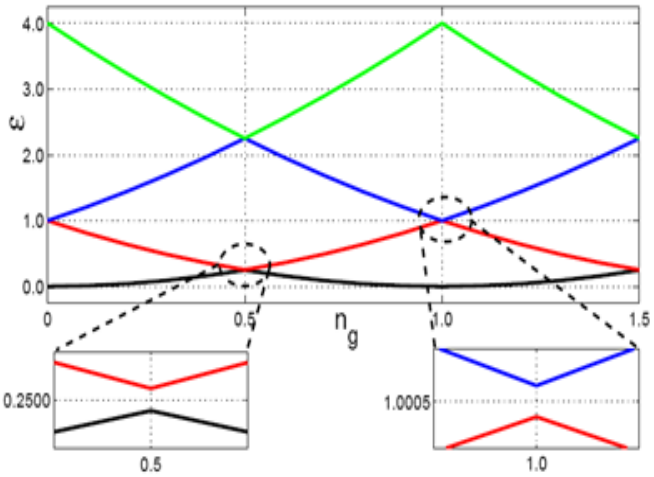

Figure 2. Eigenenergies as a function of dimensionless charge number n_g. Insets indicate the adjacent eigenenergies at n_g=0.5 and n_g=1.

Among those dimensionless charge numbers, n_g = 1/2 case is frequently called as *sweet spot* for charge qubits since the negative effects of dephasing and noise can be mostly eliminated at this operational condition where the optimal working point is also satisfied [20]. Energy difference between the leading states namely |0> and |1> (i.e at the sweet spot depicted in the inset of Fig.(2) is found to be around $2E_j/E_c$ = 0.0046469 which satisfies a well-known parametrical relation from the literature (see for example [13,14,20]). In order to show the detailed numerical results, eigenenergies with respect to eigenstates are tabulated in Table 1 for various n_gs.

Table 1. Eigenenergies for the half-integral and integral charge numbers.

Eigenstate	Eigenenergy ϵ_n	
n	$n_g = 1/2$	$n_g = 3/2$
0	0.2478002	0.2477768
1	0.2524471	0.2524781
2	2.2507520	2.2500025
3	2.2515028	2.2522697
4	6.2524953	6.2512502
5	6.2537466	6.2550289

Eigenstate	Eigenenergy ϵ_n	
n	$n_g = 0$	$n_g = 1$
0	-9.60×10^{-6}	-8.92×10^{-6}
1	1.0004996	1.0000047
2	1.0005096	1.0010083
3	4.0019988	4.0010009
4	4.0019989	4.0030080
5	9.0044853	9.0029918

In the cases of integer n_g values, the closest energy difference between eigenstates of |1> and |2> is found around Ej/(2Ec) = 0.0010036 at n_g = 1. Thus the energy difference for n_g is higher than the case for the integer n_g.

4. CONCLUSIONS

A numerical study on the eigenvalues of a superconducting circuit is realized by using the Schrodinger formalism. Using the methodology of finite difference method, the eigenenergies and corresponding eigenfunctions are sensitively determined. Consequently, we claim that our solution method for similar circuit problems can be clearly utilized to obtain the full eigenstates of the system. In addition, our approach is very useful in order to find out the initial estimation of experimental parameters in the explorations of charge qubits. This method can be compared with the real experiments determining the gate voltage-device voltage relationship, if the same experimental inputs are adjusted to the model.

[1]. *Y. Makhlin, G. Schon, and A. Shnirman*, Rev. Mod. Phys. 73 (2001) 357.

[2]. *V.K. Semenov, D.V. Averin,* IEEE Trans. Appl. Supercond. 13 (2003) 960.

[3]. *D.S. Crankshaw, J.L. Habif, X. Zhou, T.P. Orlando, M.J. Feldman, M.F.J. Bocko,* IEEE Trans. Appl. Supercond. 13 (2003) 966.

[4]. *H.J.M. ter Brake, F.-Im. Buchholz, G. Burnell, T. Claeson, D. Crete, P. Febvre, G.J. Gerritsma, H. Hilgenkamp, R. Humphreys, Z. Ivanov, W. Jutzi, M.I. Khabipov, J. Mannhart, H.-G. Meyer, J. Niemeyer, A. Ravex, H. Rogalla, M. Russo, J. Satchell, M. Siegel, H. Topfer, F.H. Uhlmann, J.-C. Villegier, E. Wikborg, D. Winkler and A.B. Zorin,* Physica C: Supercon., 439/1 (2006) 1.

[5]. *S.K.Dana, D. Sengupta and K. Edoh,* IEEE Trans. Circ.Sys.-I: Fund. Theo. Appl., 48/8 (2001) 990.

[6]. *Y. Nakamura, Y. A. Pashkin, and J. S. Tsai,* Nature, 398 (1999) 786.

[7]. *J. Matisoo,* IEEE Trans. Magnet, 5 (1969) 848.

[8]. *G. Paraoanua and A. M. Halvari*, Appl. Phys. Lett. 86 (2005) 093101.

[9]. *M. Tinkham,* Introduction to Superconductivity, McGraw-Hill, Singapore, 1996.

[10]. *R. Dolata, H. Scherer, A. Zorin, and J. Niemeyer,* Appl. Phys. Lett. 80 (2002) 2776.

[11]. *J. Sakurai and S. F. Tuan (Editor),* Modern Quantum Mechanics, The Benjamin/Cummings Publishing Company Inc., 1985.

[12]. *Yu.A. Pashkin, T.Yamamoto, O.Astafiev, Y.Nakamura, D.V.Averin, T.Tilma, F.Nori and J.S.Tsai,* Physica C 426-431 (2005) 1552.

[13]. *D. Vion, A. Aassime, A. Cottet, P. Joyez, H. Pothier, M.H. Devoret, C. Urbina and D. Esteve,* Physica E 18 (2003) 7.

[14]. *V. Bouchiat, D. Vion, P. Joyez, D. Esteve, C. Urbina and M. H. Devoret,* Appl.Supercon., 6/10-12 (1998) 491.

[15]. *M.Canturk and E.Kurt,* Phys. Scrip. 76/6 (2007) 634.

[16]. *Y.Wang and Y. Shi,* J. Math. Anal. Appl. 309 (2005) 56.

[17]. *D.J. Evans and S.O. Okolie,* Comp. & Maths. Appls. 8/2 (1982) 157.

[18]. *Ake Bjorck and Gene H. Golub,* SIAM Review 19/1 (1977) 5.

[19]. *M.G. Castellano, R. Leoni, G. Torrioli, B. Buonomo, D. Simoeone, P. Carelli, F.Chiarello, F.Mattioli and C. Comelli,* Supercond. Sci. Techn., 14 (2001), 1156-1160.

[20]. *D.Vion,* in Quantum Entanglement and Information Processing (Les Houches Session LXXIX), edited by J.-M. Raimond, J. Dalibard, and D. Esteve (Elsevier, New York, 2003), pp. 443-485.