# EXACT SOLUTION OF NON-ABELIAN CONFORMAL AFFINE TODA MODEL

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The discrete symmetry transformation method has been applied for non-abelian conformal affine Toda models.

1. Toda Field Theories can be divided into three categories; each one exhibiting nice characteristic properties. First, associated to the finite simple Lie algebras, there are the Conformal Toda models, which are conformally invariant 1+1 field theories. Even more, they permit the construction of extensions of the Virasoro algebra including higher spin generators, namely W-algebras. The second class of theories are the Affine Toda models, based on loop algebras, which can be regarded as a perturbed Conformal Toda model where the conformal symmetry is broken by the perturbation while the integrability is preserved [1]. One of their main properties is that they possess soliton solutions. These two classes of models are called abelian or non-abelian referring to whether their fields live on an abelian or non-abelian group [2, 3, 4, 5].Finally, the conformal symmetry can be restored in the abelian Affine Toda models just by adding two extra fields which do not modify the dynamics of the original model; one of these fields is a connection whose only role is to implement the conformal invariance. These are the so called Conformal Affine Toda models [6, 7], and they are based on a full Kac-Moody algebra; moreover, they are integrable [8], and have soliton solutions [9]. In fact, many properties of the Affine Toda models can be more easily understood by considering them as the Conformal Affine Toda models with the conformal symmetry spontaneously broken.

At the same time the problem of constructing of the solutions of self-dual Yang-Mills (SDYM) model and its dimensional reductions, the one dimensional WZNW model in our case, in the explicit form for arbitrary semisimple Lie algebra, rank of which is greater than two, remains important for the present time. The interest arises from the fact that almost all integrable models in one, two and (1+2)-dimensions are symmetry reductions of SDYM or they can be obtained from it by imposing the constraints on Yang-Mills potentials [10-27].

Two effective methods of generating of the exact solutions, the Riemann Hilbert Problem formalism [20] and the discrete symmetry transformation method [22], have been applied to Toda like systems. This work is devoted to construct a group theoretical background of earlier considerations.

2.The two-loop WZNW model was reduced [28] to generalized non-abelian conformal affine Toda model:

$$\partial_+(\partial_- BB^{-1}) = \left[\Lambda_{-l}, B\Lambda_l B^{-1}\right] \qquad (2.1)$$

which can also be written as

$$\partial_-(B^{-1}\partial_+ B) = -\left[\Lambda_l, B^{-1}\Lambda_{-l}B\right] \qquad (2.2)$$

On the other hand, the one dimensional reduction of self duality equations obtained in [20] are the equations for the element $f$, taking values in the semisimple algebra,

$$\frac{\partial^2 f}{\partial r^2} + 2\frac{\partial f}{\partial r} - [H,[H,f]] - 2[X^-,[X^+,f]] - 2[X^+,[X^-,f]] + 2[[\frac{\partial}{\partial r} - H, f],[X^+,f]] = 0 \qquad (2.3)$$

Here $H, X^{\pm}$ are generators of $A_1(SL(2,C))$ algebra

$$\left[X^+_M, X^-\right] = H, \left[H, X^{\pm}\right] = \pm 2X^{\pm}$$

embedded to gauge algebra in the half-integer way. Let’s rewrite (2.3) in the equivalent form:

$$[\frac{1}{2}(\frac{\partial}{\partial r} + H) - [X^+, f], -\frac{1}{2}[\frac{\partial}{\partial r} - H, f] + X^-] - \frac{1}{2}[\frac{\partial}{\partial r} - H, f] + X^- = 0$$

This equation after changing the variable $t = \ln r$ has the following form

$$[\frac{\partial}{\partial t} + \frac{1}{2}H - [X^+, f], -\frac{\partial f}{\partial t} + \frac{1}{2}[H, f] + X^-] - \frac{\partial f}{\partial t} + \frac{1}{2}[H, f] + X^- = 0 \qquad (2.4)$$

Introducing the notation

$$\tilde{F} = e^{\frac{1}{2}Ht}\left(-\frac{\partial f}{\partial t} + \frac{1}{2}[H, f] + X^{-}\right)e^{-\frac{1}{2}Ht} \quad , \qquad (2.5)$$

multiplying (2.4) from the left side by $e^{\frac{1}{2}Ht}$ and from the right side by $e^{-\frac{1}{2}Ht}$, we obtain

$$\frac{\partial \tilde{F}}{\partial t} - [[e^{\frac{1}{2}Ht} X^{+} e^{-\frac{1}{2}Ht}, e^{\frac{1}{2}Ht} f e^{-\frac{1}{2}Ht}], \tilde{F}] + \tilde{F} = 0$$

Due to the evident equality

$$e^{\frac{1}{2}Ht} X^{+} e^{-\frac{1}{2}Ht} = e^{t} X^{+}$$

the last equation can be rewritten in a form

$$\frac{\partial \tilde{F}}{\partial t} - e^{t}[[X^{+}, \tilde{f}], \tilde{F}] + \tilde{F} = 0 \ , \qquad (2.6)$$

where

$$\tilde{f} = e^{\frac{1}{2}Ht} f e^{-\frac{1}{2}Ht} .$$

In terms of these notations we have from (2.5) the following expression

$$\tilde{F} = -\frac{\partial \tilde{f}}{\partial t} + [H, \tilde{f}] + X^{-}e^{-t} = 0$$

Let's introduce the notation

$$F = e^{t}\tilde{F} = -e^{t}\frac{\partial \tilde{f}}{\partial t} + e^{t}[H, \tilde{f}] + X^{-} = 0$$

Then (2.6) has a form

$$\frac{\partial F}{\partial t} + [A, F] = 0 \quad , \qquad (2.7)$$

where $A = -e^{t}[X^{+}, \tilde{f}]$.

The equation (2.7) is one-dimensional evolution equation defined by Lax pair operators and it is one of the principal criteria of equations integrability.
From the presentation (2.5) it is followed that

$$\frac{\partial}{\partial t} spF^{n} = 0, \ for \ \forall n$$

and solution of the equations can be found in a form

$$F = \varphi F_0 \varphi^{-1}, \qquad (2.8)$$

where $\varphi(t)$ takes values in the corresponding Lie group and $F_0 = F|_{t=0}$.

From equation (2.7) and presentation (2.8) it is directly followed the expression for the operator $A$:

$$A = \varphi' \varphi^{-1} \qquad (\varphi' = \frac{\partial \varphi}{\partial t}) \qquad (2.9)$$

Let's consider the commutator of $F$ with $X^{+}$:

$$[X^{+}, F] = [X^{+}, X^{-}] - e^{t}\frac{\partial}{\partial t}[X^{+}, \tilde{f}] + e^{t}[X^{+}, [H, \tilde{f}]] =$$
$$= H - e^{t}\frac{\partial}{\partial t}[X^{+}, \tilde{f}] - 2e^{t}[X^{+}, \tilde{f}] + e^{t}[X^{+}, [H, \tilde{f}]] =$$
$$= H - \frac{\partial}{\partial t}(e^{t}[X^{+}, \tilde{f}]) - e^{t}[X^{+}, \tilde{f}] + [H, e^{t}[X^{+}, \tilde{f}]].$$

Taking into account (3.6) and (3.7) the last expression can rewritten in a form

$$[X^{+}, \varphi F_0 \varphi^{-1}] = H - (\varphi' \varphi^{-1})' - \varphi' \varphi^{-1} + [H, \varphi' \varphi^{-1}] .$$

Making the substitution $\varphi = e^{Ht}q$ and introducing a new variable $\tau = e^{-t}$, we have

$$\frac{\partial}{\partial \tau}\left(\frac{\partial q}{\partial \tau} q^{-1}\right) = [qF_0 q^{-1}, X^{+}] \qquad (2.10)$$

Equation (2.10) is one-dimensional generalized non-abelian conformal affine Toda model as it is obviously seen from eq. (2.1).

3. Let's consider the two-dimensional generalization of the eq. (2.10):

$$\frac{\partial}{\partial z}\left(\frac{\partial q}{\partial \bar{z}} q^{-1}\right) = [qF_0 q^{-1}, X^{+}] \qquad (3.1)$$

It can be shown that the equation (3.1) can be further reduced to the form

$$2\partial_z \partial_{\bar{z}} f = [f_z, f_{\bar{z}}] \qquad (3.2)$$

The next question how to obtain from this solution new solutions using the discrete symmetry transformation. For the case of algebra $A_1(SL(2,C)$, $f = f^{-}X^{-} + f^{0}H + f^{+}X^{-}$, it takes a form:

$$F^{-}=\frac{1}{f^{+}}$$
$$\frac{\partial F^{0}}{\partial z}=(f^{0}-F^{0}+1)\frac{\partial \ln f^{+}}{\partial z}-\frac{\partial f^{0}}{\partial z}$$
$$\frac{\partial F^{0}}{\partial \bar{z}}=(f^{0}-F^{0}-1)\frac{\partial \ln f^{+}}{\partial \bar{z}}-\frac{\partial f^{0}}{\partial \bar{z}} \qquad (3.3)$$
$$\frac{\partial F^{+}}{\partial z}=(f^{0}-F^{0}+1)^{2}\frac{\partial \ln f^{+}}{\partial z}-2f^{+}(f^{0}-F^{0}+1)\frac{\partial f^{0}}{\partial z}-(f^{+})^{2}\frac{\partial f^{-}}{\partial z}$$
$$\frac{\partial F^{+}}{\partial \bar{z}}=(f^{0}-F^{0}-1)^{2}\frac{\partial \ln f^{+}}{\partial \bar{z}}-2f^{+}(f^{0}-F^{0}-1)\frac{\partial f^{0}}{\partial \bar{z}}-(f^{+})^{2}\frac{\partial f^{-}}{\partial \bar{z}}$$

Here $f(f^{+},f^{0},f^{-})$ is considered to be a known solution of equation (3.2) and $F(F^{+},F^{0},F^{-})$ is one to be determined.
The equations (3.3) can be solved (integrated) completely in the case of the following initial conditions:

$$f^{0}=\begin{pmatrix}\tau & \alpha^{0}\\ 0 & -\tau\end{pmatrix}, \qquad (3.4)$$
$$\partial_{z}\partial_{\bar{z}}\tau=0,\ \partial_{z}\partial_{\bar{z}}\alpha^{0}=\partial_{z}\alpha^{0}-\partial_{\bar{z}}\alpha^{0}$$

Using the conformal invariance of the equations (3.2) we can take the solution of the equation for $\tau$ in the form $\tau=z+\bar{z}$. The result of integration of (3.4) may expressed in terms of a solutions of the following system of linear equations

$$\partial_{z}\alpha^{l}-2\alpha^{l}=\partial_{z}\alpha^{l+1},$$
$$-\partial_{\bar{z}}\alpha^{l}-2\alpha^{l}=\partial_{\bar{z}}\alpha^{l+1} \qquad (3.5)$$

and is given by

$$f_{-}^{n}=\frac{Det_{n-1}(\alpha)}{Det_{n}(\alpha)},\ f_{0}^{n}=\frac{\breve{D}et_{n}(\alpha)}{Det_{n}(\alpha)},\ f_{+}^{n}=\frac{Det_{n+1}(\alpha)}{Det_{n}(\alpha)} \qquad (3.6)$$

$(Det_{-1}=\breve{D}et_{0}=0,\ \ Det_{0}=1)$, where $Det_{n}(\alpha)$ are the minors of order n of the following matrix:

$$\alpha=\begin{pmatrix}\alpha^{0} & \alpha^{1} & \alpha^{2} & .... & ....\\ \alpha^{1} & \alpha^{2} & \alpha^{3} & .... & ....\\ \alpha^{2} & \alpha^{3} & \alpha^{4} & .... & ....\\ .... & .... & .... & .... & ....\\ .... & .... & .... & .... & ....\end{pmatrix} \qquad (3.7)$$

Here $\alpha^{l}$ are the solutions of the linear system (3.5) and $\breve{D}et_{n}(\alpha)$ denotes that in the last row of the corresponding matrix the indices of $\alpha^{l}$ have been increased by one.


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## TODUN QEYRİ-ABEL KONFORM AFİN MODELİNİN DƏQİQ HƏLLİ

Diskret simmetriya çevrilmələri metodu Todun qeyri-Abel conform afin modeli üçün tətbiq edilmişdir.

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## ТОЧНОЕ РЕШЕНИЕ НЕАБЕЛЕВОЙ КОНФОРМНОЙ АФФИННОЙ МОДЕЛИ ТОДА

Метод преобразований дискретной симметрии применен для неабелевых конформных аффинных моделей Тода.

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