THE GENERAL PROPERTIES OF NONLINEAR FORMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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The general properties of nonlinear partial differential equations of polynomial type have been revealed. The obtaining method of new solutions of nonlinear equations with the help of operator integrals of motion is shown.

In work [1] the nonlinear partial differential equations containing the derivatives of third order, derivatives of second and first orders have considered. Especially, the equations of following three types have considered:

$$\varphi_{xxx} + \Lambda(\varphi_{xx}, \varphi_{tt}, \varphi_{xt}, \varphi_t) = 0$$
$$\varphi_{xxt} + \mathrm{M}(\varphi_{xx}, \varphi_{tt} \ldots) = 0$$
$$\varphi_{xxx} + \varphi_{xtt} + \mathrm{N}(\varphi_{xx}, \varphi_{tt}) = 0$$

and symmetrically to $x \Leftrightarrow t$.

The exchanges of dependent variables with the help, of which each type of these equations is led to linear one correspondingly, are shown. In this work we will reveal the general properties of these nonlinear equations.

1. GENERAL EQUATION PROPERTIES

Let's discuss the some general properties of above mentioned equations. For definition we will consider the equation of the following type:

$$\varphi_t = \hat{F}[\varphi] \tag{1.1}$$

where $\hat{F}$ is nonlinear operator, such that $\hat{F}[\varphi]$ is the some function on x, t, φ and different derivatives. We show the operator $\hat{\mathrm{I}}$ as operator integral of motion of equation (1.1) if for some equation solution $\varphi(x,t)$ $\hat{\mathrm{I}}[\varphi]$ function is also the solution of this equation [2]. It is proved that the operator $\hat{\mathrm{I}}$ should satisfy the equation:

$$\left[\hat{\mathrm{I}}, \frac{\partial}{\partial t} - \hat{F}\right][\varphi] = \hat{\mathrm{I}}[\varphi_t] - \frac{\partial}{\partial t}\left(\hat{\mathrm{I}}[\varphi] - \hat{\mathrm{I}}\left[\hat{F}[\varphi]\right] + F\left[\hat{\mathrm{I}}[\varphi]\right] = 0\right) \tag{1.2}$$

($[\hat{\mathrm{I}}, \hat{F}]$ is commutator of $\hat{\mathrm{I}}$ and $\hat{F}$ operators) on space of (1.2) equation solution.

Let's suppose that equation (1.1) is the linear form of linear equation [3]:

$$\Psi_t = \hat{\mathrm{H}}\Psi \tag{1.3}$$

$\hat{\mathrm{H}}$ is linear operator. This means that such nonlinear operator $\hat{\kappa}$ exists that $\varphi = \hat{\kappa}[\Psi]$ is equation solution (1.1) at the condition that Ψ satisfies to equation (1.3). The motion integrals $\hat{\Lambda}$ of equation (1.3) satisfy to equation:

$$\left[\hat{\Lambda}, \frac{\partial}{\partial t} - \hat{\mathrm{H}}\right]\Psi = 0 \tag{1.2a}$$

It is obvious that there is infinite number of operators satisfying to equation (1.2a). (For example, if $\hat{\mathrm{H}} = \frac{\partial^2}{\partial x^2}$ then $\left(\frac{\partial}{\partial x}\right)^n$ operators are motion integrals for *n=0,1,2…*). However, if equation (1.3) contains *N* of independent space derivatives (in the case of Schrödinger equation this means that we consider the dynamic quantum system with N degrees of freedom) when 2N of independent motion integrals exactly exists, rest motion integrals are some functions of these 2*N*. Indeed, if operator $\hat{U}(t)$ is evolution operator of equation (1.3), i.e. $\Psi(t) = \hat{U}\Psi(0)$, then operators:

$$\hat{\mathrm{X}}_{0j} = \hat{U}\hat{\mathrm{X}}_j\hat{U}^{-1};\ j = 1,2 \ldots \mathrm{N}$$
$$\hat{\mathrm{P}}_{0j} = \hat{U}\hat{\mathrm{P}}_j\hat{U}^{-1};\ \mathrm{P}_j = \partial / \partial xj \tag{1.4}$$

are motion integrals. We consider the systems for which there is evolution operator on some interval $0 \le t \le \mathrm{T}$; in the case of Schrödinger equation it is obvious that $\mathrm{T}=\infty$. Then $\hat{U}^{-1}$ operator also exists, if we consider it on space of equation solution (1.3). Besides, Stone-fon Neumann theory confirms that all other motion integrals are functions of operators (1.4):

$$\hat{\Lambda} = f\left(\hat{X}_{01},...\hat{X}_{0N};\hat{P}_{01},...\hat{P}_{0N}\right) = \hat{U}f\left(X_1,...X_N;\frac{\partial}{\partial x_1},...\frac{\partial}{\partial x_N}\right)\hat{U}^{-1}$$

Consequently, all operator motion integrals of equation (1) have the form:

$$\hat{I} = \hat{\kappa}f\left(X_{01},...X_{0N};P_{01},...P_{0N}\right) \quad (1.5)$$

where f is arbitrary function 2N of derivatives. It is obvious that operator (1.2) not for arbitrary function $\varphi(x,t)$ but only for solution equation (1.1). Consequently, if $\varphi(x,t)$ is equation solution (1.1), then all functions of following type

$$\tilde{\varphi} = \hat{\kappa}\left[f\left(X_{01},...X_{0N};P_{01},...P_{0N}\right)\hat{\kappa}^{-1}[\varphi]\ \right] \quad (1.6)$$

are also the solutions of this equation.

2. THE NEW EQUATION SOLUTIONS

Let's consider for example, the equation being the first integral of Burgers-Hopf equation:

$$\varphi_t + \frac{1}{2}\varphi_x^2 = \mu\varphi_{xx} \quad (2.1)$$

The exchange $\Psi = exp(-\varphi/2\mu)$ transforms the equation (2.1) into thermal conduction equation $\Psi_t = \mu\Psi_{xx}$. The motion integrals of thermal conduction equation are:

$$\hat{P}_0 = \partial/\partial x \text{ and } \hat{X}_0 = X + 2\mu t\cdot\partial/\partial x$$

Consequently, we obtain the following class of transformations:

$$\tilde{\varphi} = \hat{I}[\varphi] = -2\mu\, ln\left\{ f\left(\frac{\partial}{\partial x}, X + 2\mu t\partial/\partial x\right) exp(-\varphi/2\mu)\ \right\} \quad (2.2)$$

Let's write the equation (2.2) for concrete function f:

$$f\left(\frac{\partial}{\partial x}, X + 2\mu t\partial/\partial x\right) exp(-\varphi/2\mu) = \\ = \frac{\partial}{\partial x}\left[X\,exp(-\varphi/2\mu) + 2\mu t\frac{\partial}{\partial x}exp(-\varphi/2\mu)\right] \quad (2.3)$$

Then the new solutions of equation (2.2) will be written by following form:

$$\tilde{\varphi} = \varphi - 2\mu\, ln\left[1 - \frac{x}{2\mu}\varphi_x - t\left(-\frac{1}{2\mu}\varphi_x^2 + \varphi_{xx}\right)\right] \quad (2.4)$$

The transformations of equation (2.2) haven't been known yet, though the equation (2.1) and Burgers-Hopf equation have been studied by many authors during several years.

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XÜSUSİ TÖRƏMƏLƏRDƏ XƏTTİ TƏNLİKLƏRİN QEYRİ-XƏTTİ FORMALARININ ÜMUMİ XASSƏLƏRİ

Polinomial növlü xüsusi törəmələrdə qeyri-xətti diferensial tənliklərin ümumi xassələri aydınlaşdırılmışdır. Operator hərəkət inteqrallarının köməyilə qeyri-xətti tənliklərin yeni həllərinin alınma metodu göstərilmişdir.

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ОБЩИЕ СВОЙСТВА НЕЛИНЕЙНЫХ ФОРМ ЛИНЕЙНЫХ УРАВНЕНИЙ В ЧАСТНЫХ ПРОИЗВОДНЫХ

Выявлены общие свойства нелинейных дифференциальных уравнений в частных производных полиномиального вида. Показан метод получения новых решений нелинейных уравнений с помощью операторных интегралов движения.

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