Solution of the Relativistic Dirac Equation for Woods-Saxon potential

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Abstract

The single particle Dirac equation is solved for the spherically symmetric Woods-saxon potential. Two components spinor both for up and down direction spin along with it's eigen value energy spectrum has been found. The results derived here are good agreed with other works[1] in non-relativistic limit.

Keywords: Dirac equation; spinors Woods-Saxon potential; Jacobi equation; Bound State; energy spectrum;Hamiltonian;schrödinger-like

1 Introduction

The spherical Woods-Saxon potential that was used as a major part of nuclear shell model, was successful to deduce the nuclear energy levels[2]. Also it was used as central part for the interaction of neutron with heavy nucleus [3]. With the help of the axially-deformed Woods-Saxon potential along with the spin-orbit interaction potential , we may construct the structure of single-particle shell model [4]. The Woods-Saxon potential was used as a part of optical model in elastic scattering of some ions with heavy target in low range of energies [5]. Generally, the Woods-Saxon potential and it's various modified shapes was successful to describe the metallic clusters [6]. Recently the relativistic Dirac equation has been solved using two component spinors for Woods-Saxon potential in a special case [7].By using of Woods-saxon potential It has been shown that the isospin asymmetry of the nuclear pseudo spin interaction, which has quasi-isospin symmetry, is opposed to the nuclear spin-orbit interaction [8].

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2 method for solving the Relativistic Dirac equation

The relativistic Dirac equation is a covariant first order differential equation in a four dimensional space-time representation. In The one dimensional Dirac equation , Solution can be simplified by adopting of two components approach. These components contain solution for positive and negative energy spinors. The free particle Dirac equation in unit of $\hbar = m = 1$ can be written as

$$(i\gamma^{\mu}\partial_{\mu} - \lambda^{-1})\psi = 0. \tag{1}$$

where λ is the Compton reduced wavelength $\hbar/m.c$, m is the rest mass of particle and c is the speed of light. By the summation over repeated index in four dimensional representation we have

$$\gamma^{\mu}\partial_{\mu} \equiv \sum_{\mu=0}^{3}\gamma^{\mu}\partial_{\mu} = \gamma^{0}\partial_{0} + \overrightarrow{\gamma}.\overrightarrow{\partial} = \lambda\gamma^{0}\frac{\partial}{\partial_{t}} + \overrightarrow{\gamma}.\overrightarrow{\nabla}.\{\gamma^{\mu}\}_{\mu=0}^{3} \tag{2}$$

and the four components square matrices related to this equation satisfying the following anti-commutation relation

$$\{\gamma^{\mu},\gamma^{\nu}\} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2G \tag{3}$$

where G is the metric of Minkowski Space-time representation. A set of four dimensional matrix representation corresponding to spin $\frac{1}{2}$ particles that satisfy the above anti-commutation relation are

$$\gamma^{0} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \overrightarrow{\gamma} = \begin{pmatrix} 0 & \overrightarrow{\sigma} \\ -\overrightarrow{\sigma} & 0 \end{pmatrix} \tag{4}$$

where I is the 2×2 unit matrix and $\overrightarrow{\sigma}$'s are the 2×2 hermitian Pauli matrices. By coupling the Dirac's particle to four dimensional potential $A_{\mu} = (A_0, \overrightarrow{A})$ Gauge invariant coupling is satisfied by the substitution $\partial_{\mu} \rightarrow \partial_{\mu} + i\lambda A_{\mu}$. This choice transform the free particle Dirac equation to

$$[i\gamma^{\mu}(\partial_{\mu} + i\lambda A_{\mu}) - \lambda^{-1}]\psi = 0. \tag{5}$$

where ψ is the four-component wave function. The components of this equation in terms of potential filed can be written as

$$i\lambda\gamma^{0}\frac{\partial}{\partial t}\psi = (-i\overrightarrow{\gamma}.\overrightarrow{\nabla} + \lambda\overrightarrow{\gamma}.\overrightarrow{A} + \lambda\gamma^{0}A_{0} + \lambda^{-1})\psi \tag{6}$$

by multiplying both side of this equation with $\gamma^{0}\lambda{-1}$ we may get

$$i\frac{\partial}{\partial t}\psi = (-i\lambda^{-1}\overrightarrow{\alpha}.\overrightarrow{\nabla} + \overrightarrow{\alpha}.\overrightarrow{A} + A_{0} + \lambda^{-2}\beta)\psi \tag{7}$$

where $\overrightarrow{\alpha}$ and β are defined by the following hermitian matrices

$$\overrightarrow{\alpha} = \gamma^{0}\overrightarrow{\gamma} = \begin{pmatrix} 0 & \overrightarrow{\sigma} \\ \overrightarrow{\sigma} & 0 \end{pmatrix} and \qquad \beta = \gamma^{0} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{8}$$

using equation(7) for time independent potential one can show the Dirac hamiltonian (in unit of $mc^2 = 1/\lambda^2$) by following matrix

$$H = \begin{pmatrix} \lambda^2 A_0 + 1 & -i\lambda\overrightarrow{\sigma}.\overrightarrow{\nabla} + \lambda^2\overrightarrow{\sigma}.\overrightarrow{A} \\ -i\lambda\overrightarrow{\sigma}.\overrightarrow{\nabla} + \lambda^2\overrightarrow{\sigma}.\overrightarrow{A} & \lambda^2 A_0 - 1 \end{pmatrix} \tag{9}$$

then the wave equation will be $(H - \varepsilon)\psi = 0.$, where ε is the real relativistic energy in unit of mc^2. equation (9) is invariant under the usual gauge transformation, so the hamiltonian can be replace by

$$H = \begin{pmatrix} \lambda^2 A_0 + 1 & -i\lambda\overrightarrow{\sigma}.\overrightarrow{\nabla} + i\lambda^2\overrightarrow{\sigma}.\overrightarrow{A} \\ -i\lambda\overrightarrow{\sigma}.\overrightarrow{\nabla} - i\lambda^2\overrightarrow{\sigma}.\overrightarrow{A} & \lambda^2 A_0 - 1 \end{pmatrix} \tag{10}$$

By considering spherical symmetry we can choose $(A_0, \overrightarrow{A})$ as $[V(r), \frac{1}{\lambda}\hat{r}W(r)]$, where $\hat{r}$ is the radial vector,$V(r)$ and $W(r)$ are real radial functions referred to even and odd components of relativistic potential respectively.Therefore the spinors can be write as [9]

$$\psi = \begin{pmatrix} i[g(r)/r]\chi^j_{lm} \\ (f(r)/r)\overrightarrow{\sigma}.\hat{r}\chi^j_{lm} \end{pmatrix} \tag{11}$$

where *f(r)* and*g(r)* are real square integrable functions. The angular component of the spinors ,χ^j_{lm}, can be shown by following relation

$$\chi^j_{lm} = \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \sqrt{l \pm m + 1/2}Y_l^{m-1/2} \\ \pm\sqrt{l \pm m + 1/2}Y_l^{m+1/2} \end{pmatrix}, for j = l \pm 1/2 \tag{12}$$

where $Y_l^{m\pm1/2}$ is the spherical harmonic function , m is an integer quantum number define in the range $-j, -(j+1), -(j+2), ...(j-1), j$ and should not be confused with the mass. For the system having spherical symmetry $i\overrightarrow{\sigma}.(\overrightarrow{r} \times \overrightarrow{\nabla})\psi(r,\hat{r}) = -(1+K)\psi(r,\hat{r})$, where K is the spin orbit quantum number and define in the range $K = \pm(j + 1/2) = \pm1, \pm2, ...$ for $l = j \pm 1/2$. using this, one can show the tensor elements in hamiltonian by following relations

$$\begin{aligned} (\overrightarrow{\sigma}.\overrightarrow{\nabla})F(r)\chi^j_{lm} &= \left(\frac{dF}{dr} + \frac{1+K}{r}F\right)\chi^j_{lm} \\ (\overrightarrow{\sigma}.\overrightarrow{\nabla})(\overrightarrow{\sigma}.\hat{r})F(r)\chi^j_{lm} &= \left(\frac{dF}{dr} + \frac{1+K}{r}F\right)(\overrightarrow{\sigma}.\hat{r})\chi^j_{lm} \end{aligned} \tag{13}$$

By substituting these in the wave equation we get

$$\begin{pmatrix} +1 + \lambda^2 V(r) - \varepsilon & \lambda[\frac{K}{r} + W(r) - \frac{d}{dr}] \\ \lambda[\frac{K}{r} + W(r) + \frac{d}{dr}] & -1 + \lambda^2 V(r) - \varepsilon \end{pmatrix} \begin{pmatrix} g(r) \\ f(r) \end{pmatrix} = 0. \tag{14}$$

this matrix equation generate a couple of first order differential equations that should be solved together to get radial parts of spinors. By eliminating one component we can get a second order differential equation for another one. In absence of potential,these equations will not be schrödinger like, because contain first derivative in presence of second one. To obtain a schrödinger like equation we may make a transformation. In order to do this a global unitary transformation,$u(\eta) = exp(\frac{1}{2}\lambda\eta\sigma_2)$, is applied to relation (14)with η and σ_2 as a real constant parameter and Pauli matrix for spin $\frac{1}{2}$ particles respectively. It is also necessary to define $V(r) = \xi[W(r) + \frac{K}{r}]$,with ξ as a real parameter and $\sin(\lambda\eta) = \pm\lambda\xi$, to get schrödinger like equation where $-\frac{\pi}{2} < \lambda\eta > +\frac{\pi}{2}$.

The unitary transformation along with other constrains applied on potential leads to following equation for even component of it,

$$\begin{pmatrix} C - \varepsilon + (1 \pm 1)\lambda^2 V & \lambda(\pm\xi + \frac{C}{\xi}V - \frac{d}{dr}) \\ \lambda(\pm\xi + \frac{C}{\xi}V + \frac{d}{dr}) & -C - \varepsilon + (1 \mp 1)\lambda^2 V \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix} = 0. \tag{15}$$

where $C = \cos(\lambda\eta) = \sqrt{1 - (\lambda\xi)^2} > 0$. and

$$\begin{pmatrix} \phi^+ \\ \phi- \end{pmatrix} = u\psi = \begin{pmatrix} \cos(\frac{\lambda\eta}{2}) & \sin(\frac{\lambda\eta}{2}) \\ -\sin(\frac{\lambda\eta}{2}) & cos(\frac{\lambda\eta}{2}) \end{pmatrix} \begin{pmatrix} g \\ f \end{pmatrix} \tag{16}$$

equation (15) leads to the following relation for spinors

$$\phi^{\mp}(r) = \frac{\lambda}{C \pm \varepsilon}\left[-\xi \pm \frac{C}{\xi}V(r) + \frac{d}{dr}\right]\phi^{\pm}(r) \tag{17}$$

and finally by applying all above said simplifications one may derive[10] the following important schrödinger -like one differential wave equation

$$\left[-\frac{d^2}{dr^2} + \frac{C}{\xi}^2 V^2 \mp \frac{C}{\xi}\frac{dV}{dr} + 2\varepsilon V - \frac{\varepsilon^2 - 1}{\lambda^2}\right]\phi^{\pm}(r) = 0. \tag{18}$$

in the next section we apply this approach to solve the relativistic Dirac equation for Woods-Saxon potential.

3 Solution of The Woods-Saxon Potential

The Woods-Saxon potential in atomic units can be written as

$$V(r) = \frac{-V_0}{1 + e^{+\omega r}} \tag{19}$$

where ω is a constant related to nuclear properties. In order to solve equation (18) for woods-saxon potential, we may choose new variable x, by $\tanh(\omega r) = \frac{e^{\omega r}-1}{e^{\omega r}+1} = x$. To change differential parts in equation(18), we need

$$\begin{aligned}\frac{d}{dr} &= \frac{\omega}{2}(1-x^2)\frac{d}{dx},\\ \frac{d^2}{dr^2} &= -\frac{\omega^2}{2}x(1-x^2)\frac{d}{dx}+\frac{\omega^2}{4}(1-x^2)^2\frac{d^2}{dx^2}\end{aligned} \tag{20}$$

by definition $\rho = -\frac{C}{\xi V_0}$ and substitution for $\frac{d}{dr}, \frac{d^2}{dr^2}$and $V(r)$ in wave equation (18) results

$$\left[\frac{\omega^2}{2}x(1-x^2)\frac{d}{dx}-\frac{\omega^2}{4}(1-x^2)^2\frac{d^2}{dx^2}+\frac{\rho(\rho\mp\omega)}{4}(1-x^2)+\frac{\rho\omega\mp 2\varepsilon V_0}{2}(1-x)-\frac{\varepsilon^2-1}{\lambda^2}\right]\phi^{\pm}(r)=0. \tag{21}$$

and dividing both side of this equation by $-\frac{\omega^2}{4}(1-x^2)$ we have

$$\left[(1-x^2)\frac{d^2}{dx^2}-2x\frac{d}{dx}-\frac{\rho(\rho\mp\omega)}{\omega^2}\frac{(1-x)}{(1+x)}-2(\frac{\rho\omega\mp 2\varepsilon V_0}{2})\frac{1}{1+x}+(\frac{\varepsilon^2-1}{\lambda^2})(\frac{4}{\omega^2})(\frac{1}{1-x^2})\right]\phi^{\pm}(x)=0. \tag{22}$$

by defining $\phi^{\pm}(x) = C_n^{\pm}u(x)P_n^{\alpha,\beta}(x)$ and differentiating relative to x we get

$$\begin{aligned}\phi^{\pm'} &= C_n^{\pm}u_n^{'}P_n + C_n^{\pm}u_nP_n^{'},\\ \phi^{\pm''} &= C_n^{\pm}u_n^{''}P_n + C_n^{\pm}u_nP_n^{'} + 2C_n^{\pm}u_n^{'}P_n^{'}.\end{aligned} \tag{23}$$

substitute these in wave equation (22) and divide resultant by $C_n^{\pm}u(x)$ to get

$$\begin{aligned}&(1-x^2)C_n^{\pm}P_n^{''}(x)+[(1-x^2)\frac{2u^{'}(x)}{u(x)}-2x]C_n^{\pm}P_n^{'}(x)\\ &+\Bigg\{[(1-x^2)\frac{u^{''}(x)}{u(x)}-2x\frac{u^{'}(x)}{u(x)}-\frac{\rho(\rho\mp\omega)}{\omega^2}(\frac{1-x}{1+x})-2(\frac{\rho\omega\mp 2\varepsilon V_0}{\omega^2})(\frac{1}{1+x})\Bigg]\\ &+\left[(\frac{\varepsilon^2-1}{\lambda^2})(\frac{4}{\omega^2})(\frac{1}{1-x^2})]\right\}C_n^{\pm}P_n(x)=0.\end{aligned} \tag{24}$$

the standard jacubi equation can be given[11] as

$$(1-x^2)P_n^{''}(x)-[\alpha-\beta+(\alpha+\beta+2)x]P_n^{'}(x)+n(\alpha+\beta+n+1)P_n(x)=0. \tag{25}$$

comparing equations (24) and (25) for coefficient of $P_n^{'}(x)$ we get

$$\begin{aligned}(1-x^2)\frac{2u^{'}}{u}-2x &= [\alpha-\beta+(\alpha+\beta+2)x],\\ (1-x^2)\frac{2u^{'}}{u} &= [\alpha(1+x)-\beta(1-x)],\\ \frac{2u^{'}}{u} &= -\frac{\alpha}{1-x}+\frac{\beta}{1+x}\end{aligned} \tag{26}$$

and after integrating we have

$$u(x) = (1-x)^{\frac{\alpha}{2}}(1+x)^{\frac{\beta}{2}} \tag{27}$$

also by equating coefficients of $P_n(x)$ we get the following conditions

$$(1-x^2)\frac{u^{''}(x)}{u(x)} - 2x\frac{u^{'}}{u} - \frac{\rho(\rho\mp\omega)}{\omega^2}(\frac{1-x}{1+x}) - 2(\frac{\rho\omega\mp 2\varepsilon V_0}{\omega^2})\frac{1}{1+x} + \frac{\varepsilon^2-1}{\lambda^2}\frac{4}{\omega^2}\frac{1}{1-x^2} = n(\alpha+\beta+n+1) \quad (28)$$

we can calculate $\frac{u^{'}(x)}{u(x)}$ and $\frac{u^{''}(x)}{u(x)}$ to get the following relations

$$\begin{aligned} \frac{u^{'}(x)}{u(x)} &= -\frac{\alpha}{2}(1-x)^{\frac{\alpha}{2}-2} + \frac{\beta}{2}(1+x)^{\frac{\beta}{2}-2}, \\ \frac{u^{''}(x)}{u(x)} &= \frac{\alpha}{2}(\frac{\alpha}{2}-1)\frac{1}{(1-x)^2} - \frac{\alpha\beta}{2}\frac{1}{1-x^2} + \frac{\beta}{2}(\frac{\beta}{2}-1)\frac{1}{(1+x)^2}. \end{aligned} \quad (29)$$

and by substituting for $\frac{u^{'}}{u}$ and $\frac{u^{''}(x)}{u(x)}$ in conditions (28) we have

$$\begin{aligned} &\frac{\alpha}{2}(\frac{\alpha}{2}-1)(1+2x+x^2) + \frac{\beta^{\pm}}{2}(\frac{\beta^{\pm}}{2}-1)(1-2x+x^2) \\ &+\alpha(x-x^2) - \beta^{\pm}(x-x^2 - \frac{\rho(\rho\mp\omega)}{\omega^2}(1-2x-x^2) - 2(\frac{\rho\omega\mp 2\varepsilon V_0}{\omega^2})(1-x) \\ &-\frac{\alpha\beta^{\pm}}{2}(1-x^2) + \frac{4(\varepsilon^2-1)}{\lambda^2\omega^2} - \frac{\alpha\beta^{\pm}}{2} = n(\alpha+\beta^{\pm}+n+1)(1-x^2) \end{aligned} \quad (30)$$

equating coefficients of x,x^2 and constants in two sides of this equation we get following relations for β^+ , β^- and εrespectively

$$\begin{aligned} \frac{\alpha}{2}(\frac{\alpha}{2}-1) + \frac{\beta^+}{2}(\frac{\beta^+}{2}-1) - \frac{\rho(\rho-\omega)}{\omega^2} - 2(\frac{\rho\omega-2\varepsilon V_0}{\omega^2}) + \frac{4(\varepsilon^2-1)}{\lambda^2\omega^2}\frac{\alpha\beta^+}{2} &= n(\alpha+\beta^++n+1) \\ \alpha(\frac{\alpha}{2}-1) - \beta^+(\frac{\beta^+}{2}-1) + \alpha + \beta^+ + 2\frac{\rho(\rho-\omega)}{\omega^2} + 2(\frac{\rho\omega-2\varepsilon V_0}{\omega^2}) &= 0 \\ \frac{\alpha}{2}(\frac{\alpha}{2}-1) + \frac{\beta^+}{2}(\frac{\beta^+}{2}-1) + \alpha + \beta^+ - \frac{\rho(\rho-\omega)}{\omega^2} + \frac{\alpha\beta^+}{2} &= -n(\alpha+\beta^++n+1) \end{aligned} \quad (31)$$

and

$$\begin{aligned} \frac{\alpha}{2}(\frac{\alpha}{2}-1) + \frac{\beta^-}{2}(\frac{\beta^-}{2}-1) - \frac{\rho(\rho+\omega)}{\omega^2} - 2(\frac{\rho\omega+2\varepsilon V_0}{\omega^2}) + \frac{4(\varepsilon^2-1)}{\lambda^2\omega^2}\frac{\alpha\beta^-}{2} &= n(\alpha+\beta^-+n+1) \\ \alpha(\frac{\alpha}{2}-1) - \beta^-(\frac{\beta^-}{2}-1) + \alpha + \beta^- + 2\frac{\rho(\rho+\omega)}{\omega^2} + 2(\frac{\rho\omega+2\varepsilon V_0}{\omega^2}) &= 0 \\ \frac{\alpha}{2}(\frac{\alpha}{2}-1) + \frac{\beta^-}{2}(\frac{\beta^-}{2}-1) + \alpha + \beta^- - \frac{\rho(\rho+\omega)}{\omega^2} + \frac{\alpha\beta^-}{2} &= -n(\alpha+\beta^-+n+1) \end{aligned} \quad (32)$$

by solving two above sets of equations we derive ε and $\beta^{\pm}$ as follow

$$\begin{aligned} \frac{\varepsilon^2-1}{\lambda^2}\frac{4}{\omega^2} &= -\alpha^2, \\ \varepsilon &= (1-\frac{\omega^2\lambda^2}{4}\alpha^2)^{1/2} \end{aligned} \quad (33)$$

and

$$\begin{aligned}
\beta^{+} &= \frac{-1}{2n+\alpha+1}\left[\alpha(\alpha+1)+2n(\alpha+n+1)+2(\frac{\rho\omega-2\varepsilon V_0}{\omega^2})\right] \\
\beta^{-} &= \frac{-1}{2n+\alpha+1}\left[\alpha(\alpha+1)+2n(\alpha+n+1)+2(\frac{\rho\omega+2\varepsilon V_0}{\omega^2})\right]
\end{aligned} \tag{34}$$

where β^+ and β^-stand for the $\pm$ spinors respectively. Then the eigen wave function can be written as

$$\begin{aligned}
\phi^{\pm}(x) &= C_n^{\pm} u(\tanh\frac{\omega r}{2})P_n^{\alpha,\beta^{\pm}}(\tanh\frac{\omega r}{2}) or \\
\phi^{\pm}(x) &= C_n^{\pm}(\frac{2}{1+e^{\omega r}})^{\frac{\alpha}{2}}(\frac{2e^{\omega r}}{1+e^{\omega r}})^{\frac{\beta}{2}}P_n^{\alpha,\beta^{\pm}}(\tanh\frac{\omega r}{2}), \\
\varepsilon &= (1-\frac{\omega^2\lambda^2}{4}\alpha^2)^{1/2}
\end{aligned} \tag{35}$$

and in the limit $\lambda \longrightarrow 0$ when $\frac{\omega^2\lambda^2}{4}\alpha^2 \ll 1$. one can expand the ε and get

$$\varepsilon \simeq 1-\frac{\omega^2\lambda^2}{8}\alpha^2 \tag{36}$$

for non-relativistic situation [16] we have

$$\varepsilon \simeq 1+\lambda^2 E \tag{37}$$

and comparing these equations for ε lead to

$$E = -\frac{\omega^2\alpha^2}{8} \tag{38}$$

for Woods-Saxon super potential in non-relativistic case one get [13]

$$E(\alpha,m) = -\frac{\hbar^2}{8ma^2}(\alpha+m)^2 \tag{39}$$

in the non-super and non-relativistic case where $m=0.$, $\hbar=m=1.$ and $\frac{1}{a}=\omega$ our results immediately match the results obtained by others[14]

4 Conclusion

In this paper it has been shown that the Woods-Saxon potential can be solve for relativistic particle. Also the eigen value energy and both lower and upper spin wave spinors has been derive in terms of jacubi series successfully. The results obtained here are good agreed with non-relativistic limit[15] that obtained recently by using of super symmetry method for extended super potential [15].

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