REEXAMINATION A TIME-DEPENDENT LINEAR POTENTIAL IN THE QUANTUM MECHANICS

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We reexamine the complete solutions of the Schrödinger equation for a particle with time-dependent mass moving in a time-dependent linear potential on the base of the evolution operator method. We solve the problem in both, configuration and momentum spaces. Appropriately choosing the initial wave functions we can obtain from the representation $\psi(t)=U(t)\psi(0)$ all kinds of wave functions of the system under consideration, in particular, those solutions which are known in the literature. For example, it is shown that evolution operator can be used to obtain the Gaussian-type, Airy-type and oscillator-type wave-packet solutions of the time-dependent system. The explicit form for the inital momentum and coordinate operators (two linear independent invariants) $\hat{p}_0(t)$ and $\hat{x}_0(t)$ are found. We show that the problem of a particle moving in a linear potential is unitary equivalent to that of a free particle.

Keywords: Time-dependent linear potential, evolution operator, invariants
PACS: 03.65.-w, 03.65.Fd, 02.30.Tb

1. INTRODUCTION

During the past several decades the analytical solutions of the Schrödinger equation with the time-dependent linear potential have attracted much attention of physicists [1-8]. To study the time-dependent quantum systems there are many methods, such as LR invariant method [9, 12], path-integral method [10], space-time transformations method [5], evolution operator method [11], etc. For instance, in Ref. [1] using the Feynman's path-integral method the solution of the time-dependent linear potential problem in the form of the Airy function was presented and was shown that the Airy packet propagates without change of form. The Wigner function and exact transition amplitude between energy eigenstates for a particle in a general time-dependent linear potential was calculated in Ref. [3]. In Ref. [4], Guedes with the help of the LR invariant method solved the time-dependent Schrödinger equation for the linear potential of the particular form $V(x,t)= qx(\varepsilon_0+ \varepsilon\cos\omega t)$. Feng [5] followed the space-time transformations of the Schrödinger equation and found plane-wave type and the Airy-packet type solutions. Later Luan et al. [6] used a non-Hermitian linear LR invariant to obtain Gaussian-type wave-packet solutions of the system. Bekkar et al. [7] gave a general solution of the Schrödinger equation with the time-dependent linear potential, which corresponds to the linear LR invariant $I_1 = A(t)\hat{p}+B(t)\hat{x}+C(t)$.

The purpose of the present paper is to undertake a completely analytical solution for the problem above by means of the evolution operator method. This method has long time been used
to solve problems in quantum mechanics and quantum field theory. We demonstrate that the evolution operator method allows us to find, in principle, all (infinitely many) solutions of this problem, including those solutions which are known in the literature [1-7]. Therefore, it can be argued that all known solutions [1-7] are in fact partial solutions to the problem under consideration. We show that a complete set of Lewis-Riesenfeld (LR) invariants for this problem is not limited to linear and quadratic invariants.The reason for this may formulate as follows: according to the evolution operator method, the solutions of the time-dependent Schrödinger equation $\hat{S}(t)\psi(t)=0$ can be represented as $\psi(t)=U(t)\psi(0)$, where $\hat{S}(t)=i\hbar\partial_t - H(t)$ and $\psi(0)$ is any function (initial wave function). The evolution operator $U(t)$ satisfies the Schrödinger equation $\hat{S}(t)U(t)=0$ with the initial condition $U(0)=1$. One can expand the function $\psi(0)$ over some complete set of the orthogonal functions $\{\psi_n(0)\}$: $\psi(0)=\sum_n c_n\psi_n(0)$. Then the wave function at arbitrary time t can be given as $\psi(t)=\sum_n c_n\psi_n(t)$, where $\psi_n(t)=U(t)\psi_n(0)$. (In the case of expansion in the Fourier integral $\psi(0)=\int g(\lambda)e^{i\lambda x}d\lambda$ for the wave function at time t we obtain an expression $\psi(t)=\int g(\lambda)\psi_\lambda(t)d\lambda$, where $\psi_\lambda(t)=U(t)e^{i\lambda x}$.)

It is well known [12] also that one can construct two (for one-dimensional system) linearly independent simple invariants $\hat{p}_0(t)$ and $\hat{x}_0(t)$, provided that the evolution operator for a quantum system exists:

$$\hat{p}_0(t)=U(t)\hat{p}U^{-1}(t),$$
$$\hat{x}_0(t)=U(t)\hat{x}U^{-1}(t) \quad (1.1)$$

They are the operators of initial momentum and coordinate. All other invariants can be expressed in terms of these operators. Recall that the invariant $I(t)$ is the operator which should commute with the Schrödinger

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operator $[\hat{S}(t), I(t)] = 0$, yielding the analogous to (1.1) expression for *I(t)*:

$$I(t) = U(t)I(0)U^{-1}(t) \quad (1.2)$$

It is clear that if $I(0) = G(\hat{p}, \hat{x})$, then $I(t) = G(\hat{p}_0(t), \hat{x}_0(t))$.

On the other hand, according to the LR invariant method [9], the solutions of the time –dependent Schrödinger equation can be constructed in terms of the eigenstates $\varphi_n(t)$ of the LR invariant $I(t)$ with the time-independent eigenvalues λ_n: $I(t)\varphi_n(t) = \lambda_n\varphi_n(t)$. The function $\varphi_n(t)$ does not satisfies the Schrödinger equation, but it is an eigenfunction of the operator $\hat{S}(t)$: $\hat{S}(t)\varphi_n(t) = s_n(t)\varphi_n(t)$. A solution of the Schrödinger equation is chosen as

$$\psi_n(t) = e^{i\alpha_n(t)}\varphi_n(t), \quad (1.3)$$

where the phase $\alpha_n(t)$ is a function of time only. It follows from the Schrödinger equation for $\psi_n(t)$ that $\alpha_n(t)$ satisfies the relation

$$\alpha_n(t) = \hbar^{-1}\int_0^t s_n(t')dt' \quad (1.4)$$

One can obtain the state (1.3) also from the eigenstate $\varphi_n(0)$ of the operator *I(0)* with the same eigenvalue λ_n by means of the evolution operator $U(t)$:

$$\psi_n(t) = U(t)\varphi_n(0) = e^{i\alpha_n(t)}\varphi_n(t). \quad (1.5)$$

Thus, the evolution operator $U(t)$ transforms any eigenstate of *I(0)* into an eigenstate of $I(t)$, or, more precisely, into a solution of the Schrödinger equation.

The initial wave function $\psi(0)$ can be expand over the complete set of the eigenfunctions $\{\varphi_n(0)\}$ of the operator *I(0)*, thereby the solution of the Schrödinger equation is obtained as:

$$\psi(t) = \sum_n c_n U(t)\varphi_n(0) = \sum_n c_n e^{i\alpha_n(t)}\varphi_n(t). \quad (1.6)$$

The solution, obtained in [7], corresponds to an expansion over the eigenfunctions of the linear invariant $I_1(0) = A_0\hat{p} + B_0\hat{x} + C_0$ at $B_0 = 0$, i.e. over the plane waves, which can be understood as a usual Fourier transformation.

It is clear that there may exist other operators *I(0)*, with complete set of the eigenfunctions, and one can expand $\psi(0)$ over this complete set. One of such kind operator is

$$I_3(0) = A_0\cosh\left(\frac{a\hat{p}}{\hbar}\right) + B_0 x(x+ia)e^{-\frac{a\hat{p}}{\hbar}}, \quad (1.7)$$

which is Hamiltonian of the relativistic linear harmonic oscillator [14,15]. The eigenfunctions of $I_3(0)$ are expressed through the Meixner-Pollaczek polynomials

$$\varphi_n(0) = N_n A_0^{-ix/a}\Gamma(\nu - ix/a)P_n^{\nu}(x/a;\pi/2), \quad (1.8)$$

where $\nu = \frac{1}{2} + \sqrt{\frac{1}{4} + A_0^2}$, $a = (2A_0B_0)^{-1/2}$. The operators $K_0 = I_3(0)$, $K_1 = -x/a$ and $K_2 = -I_3(0) + A_0\exp\left(\frac{a\hat{p}}{\hbar}\right)$ form the Lee algebra*su* (1.1), i.e. $[K_0, K_1] = iK_2, [K_1, K_2] = -iK_0$.

Thus, now it becomes clear that the complete solution of the Schrodinger equation with the time-dependent linear potential is not exhausted by the results of [1-8]. We will consider, as in [5], a more general case, i.e., a particle with time-dependent mass moving in the time-dependent linear potential. This time-dependent dynamical problem could be solved in either configuration or momentum space. It can be found that the all known in the literature solutions [1-8] are merely the particular cases in comparison with our result. We also note that in the evolution operator method there is no further problem of finding time-dependent phase, inherent LR invariant method. The derivation of the exact wave functions is straightforward and is obtained with much less effort than other results [1-7] based on the other methods.

The main results of this paper are as follows. First, we give an explicit form of the evolution operator $U(t)$ in the *x*- and *p*-representations for the Schrödinger equation describing the motion of a particle with the time-dependent mass in the time-dependent linear potential Sec. II). Second, we show that all known solutions can be derived from a general representation for the wave function $\psi(t) = U(t)\psi(0)$ (Sects. II, III, IV). Since the time-dependent system in the initial time can be in any state, the corresponding Schrodinger equation has infinitely many solutions.

However, appropriately choosing an initial wave function, one can always construct a solution of the Schrodinger equation with the required properties. For example, in Sec. V we obtained the square-integrable oscillator-like solutions. Third, we find the explicit form of the initial momentum and the initial coordinate operators $\hat{p}_0(t)$ and $\hat{x}_0(t)$, through of which all other invariants can be expressed (Sec. III).

We show that the complete set of the LR invariants for the system under consideration is not restricted by the linear and quadratic invariants (Sec. VI).

Fourth, we have shown that a problem of a particle that moves in a linear potential and a free particle problem are unitarily equivalent.

2. CONFIGURATION SPACE

The Schrödinger equation for describing the motion of a particle with time-dependent mass in the presence of time -dependent linear potential is of the from

$$i\hbar\partial_t\psi(x,t)=\left[-\frac{\hbar^2}{2M(t)}\partial_x^2-F(t)x\right]\psi(x,t) \tag{2.1}$$

where $M(t)$ and $F(t)$ are arbitrary time-dependent functions. The solution of the equation (2.1) may be obtained from the evolution operator $U(x,t)$

$$\psi(x,t)=U(x,t)\psi(x,0). \tag{2.2}$$

The explicit form of the operator $U(x,t)$ was found in [15]

$$U(x,t)=e^{\frac{ix\delta(t)}{\hbar}}e^{-\frac{i}{\hbar}\int_0^t\frac{1}{2M(t')}[-i\hbar\partial_x+\delta(t')]^2dt'}, \tag{2.3}$$

in which the notation $\delta(t)=\int_0^t F(t')dt'$ is used. Now taking into account (2.3) in (2.2), one gets a following general representation for the solution of the Schrodinger equation (2.1)

$$\psi(x,t)=e^{\frac{i}{\hbar}[x\delta(t)-s_0(t)]}e^{-s_1(t)\partial_x}e^{i\hbar s_2(t)\partial_x^2}\psi(x,0), \tag{2.4}$$

where $s_i(t)$ (i=0, 1, 2) are defined, respectively, as

$$s_0(t)=\int_0^t\frac{\delta^2(t')}{2M(t')}dt',\qquad s_1(t)=\int_0^t\frac{\delta(t')}{M(t')}dt',$$

$$s_2(t)=\int_0^t\frac{dt'}{2M(t')}. \tag{2.5}$$

If we set $M(t)=m$, we will find that

$$s_0(t)=\frac{\delta_2(t)}{2m},\qquad s_1(t)=\frac{\delta_1(t)}{m},$$

$$s_2(t)=\frac{t}{2m}, \tag{2.6}$$

Where
$$\delta_1(t)=\int_0^t dt'\int_0^{t'}F(t'')dt''=\int_0^t\delta(t')dt',$$

$$\delta_2(t)=\int_0^t dt'\left[\int_0^{t'}F(t'')dt''\right]^2=\int_0^t\delta^2(t')dt'.$$

A particular form of the evolution operator $U(x,t)$, when $M(t)=m$ and $F(t)=F_0$ was given in [8]. In this case we have $\delta(t)=F_0t$, $\delta_1(t)=F_0t^2/2$, $\delta_2(t)=F_0^2t^3/3$. Now choosing in (2.4) different initial wave functions $\psi(x,0)$ one can construct different wave functions $\psi(x,t)$ at time $t>0$.

For example, appropriately choosing the initial wave function we obtained from (2.4) all known in the literature solutions [1-8] of the equation (2.1) as the special cases:

1) $\psi(x,0)=N$. In this case we easily get

$$\psi(x,t)=Ne^{\frac{i}{\hbar}[x\delta(t)-s_0(t)]}, \tag{2.7}$$

where N is a normalization constant. To compare with the solution in Ref. [4], we let $F(t)$ take the form $-q(\varepsilon_0+\varepsilon\cos\omega t)$, and set $M(t)=m$, which yields

$$\delta(t)=-\frac{q}{\omega}(\varepsilon_0\omega t+\varepsilon\sin\omega t),$$

$$s_0(t)=\frac{q^2}{2m\omega^3}\left[\frac{\varepsilon_0(\omega t)^3}{3}+2\varepsilon\varepsilon_0(\sin\omega t-\omega t\cos\omega t)+\frac{1}{2}\varepsilon^2(\omega t-\frac{1}{2}\sin 2\omega t)\right]. \tag{2.8}$$

Substituting these expressions in (2.7), we obtain the solution of (18) in [4]

$$\psi(x,t)=N\exp\left[-\frac{iq}{\hbar\omega}(\varepsilon_0\omega t+\varepsilon\sin\omega t)x\right]\times$$
$$\times\exp\left\{-\frac{iq^2}{2m\hbar\omega^3}\left[\frac{\varepsilon_0(\omega t)^3}{3}+2\varepsilon_0\varepsilon(\sin\omega t-\omega t\cos\omega t)+\frac{1}{2}\varepsilon^2(\omega t-\frac{1}{2}\sin 2\omega t)\right]\right\}. \tag{2.9}$$

2) $\psi(x,0)=e^{iAx}/\sqrt{2\pi}$, where A is an arbitrary real number. In this case using (A.3) one can obtain the following expression for the wave function

$$\psi(x,t)=\frac{1}{\sqrt{2\pi}}e^{iA[x-s_1(t)]}e^{-i\hbar s_2(t)A^2}e^{\frac{i}{\hbar}[x\delta(t)-s_0(t)]}\quad, \tag{2.10}$$

which coincides with the formula (6) in [5].

3) $\psi(x,0)=Ai(Bx)$, where B is an arbitrary constant and $Ai(x)$ denotes the Airy function. In this case, after some simple transformations in (2.4) with the help of the formula (A6) can be shown that

$$\begin{aligned}\psi(x,t)&=e^{\frac{i}{\hbar}[x\delta(t)-s_0(t)]}e^{i\hbar s_2(t)B^3[x-s_1(t)]-\frac{2}{3}i\hbar^3 s_2^3(t)B^6}\times\\&\times Ai\left(B\left[x-s_1(t)-\hbar^2 s_2^2(t)B^3\right]\right).\end{aligned} \tag{2.11}$$

This result is equivalent to formula (8) of Ref. [5].

Note that the formula (15) of Ref. [1] [17] corresponds to the following initial condition $\psi(x,0)=Ai\left(Bx/\hbar^{2/3}\right)$ and, therefore, it is obtained from (2.11) by replacing $B\to B/\hbar^{2/3}$ and $M(t)\to m$. By choosing the initial wave function in the form $\psi(x,0)=Ai\left(-\left[2mF_0/\hbar^2\right]^{1/3}\left[x+E/F_0\right]\right)$ one gets the formula (24) of Ref. [3].

In the next two Sections we obtain a general solution of the Schrodinger equation (2.1) by the evolution operator method, which yields the results of [6, 7] as a particular cases.

3. INVARIANTS FOR THE TIME-DEPENDENT LINEAR POTENTIAL

Knowledge of the evolution operator (2.3) allows us to find not only the wave functions, but also to construct the LR invariants for the system. In the case of the time-dependent linear potential we have the following general expressions for the invariants (1.1)

$$\begin{aligned}\hat p_0(t)&=\hat p-\delta(t),\\ \hat x_0(t)&=\hat x-2s_2(t)\hat p+2\delta(t)s_2(t)-s_1(t).\end{aligned} \tag{3.1}$$

All other invariants are expressed through them. For example,

$$I_1(t)=A_0\hat p_0(t)+B_0\hat x_0(t)+C_0\equiv A(t)\hat p+B(t)\hat x+C(t), \tag{3.2a}$$

$$\begin{aligned}I_2(t)&=D_0^2\hat p_0^2(t)+K_0\hat x_0^2(t)\equiv\\&\equiv D(t)\hat p^2+E(t)(\hat p\hat x+\hat x\hat p)+K(t)\hat x^2+A'(t)\hat p+B'(t)\hat x+C'(t),\end{aligned} \tag{3.2b}$$

$$I_3(t)=A_0\cosh\left(\frac{a\hat p_0(t)}{\hbar}\right)+B_0\hat x_0(t)\left[\hat x_0(t)+ia\right]e^{-\frac{a\hat p_0(t)}{\hbar}}. \tag{3.2c}$$

At $M(t)=m$ the invariants constructed in Refs.[4, 6, 7] are obtained from (3.1) and (3.2a).

a) Let us find with the help of the evolution operator a particular solution of the Schrodinger equation (2.1), corresponding to the linear invariant (3.2a) at $B_0=0$. For this purpose, as the initial wave function we choose the eigenvector of the operator $I_1(0)$ at $B_0=0$ corresponding to the eigenvalue λ, which has the form

$$\varphi_\lambda(x,0)=e^{\frac{i}{\hbar}\lambda_1 x},\ \lambda_1=(\lambda-C_0)/A_0. \tag{3.3}$$

From (2.3), (2.4) and (3.3), we then get the solution of the equation (2.1):

$$\psi_\lambda(x,t)=U(x,t)e^{\frac{i}{\hbar}\lambda_1 x}=\exp\left\{-\frac{i}{\hbar}\int_0^t\frac{1}{2M(t')}\left[\frac{\lambda-C(t')}{A_0}\right]^2dt'\right\}\exp\left[\frac{i}{\hbar}\frac{\lambda-C(t)}{A_0}x\right], \tag{3.4}$$

where $C(t) = -A_0\delta(t) + C_0$. To derive the relation (3.4) we have used the formula

$$f(-i\hbar\partial_x)e^{\frac{i}{\hbar}\lambda_1 x} = f(\lambda_1)e^{\frac{i}{\hbar}\lambda_1 x} \quad . \tag{3.5}$$

If we set $M(t) = m$, we will find that (3.4) coincides with the formula (12) of Ref . [7] [17].

b) We now find the general solution of equation (2.1) corresponding to the linear invariant (3.2 a) at $B_0 = 0$. To this end we expand the initial wave function over the plane waves (3.3), i.e.

$$\psi(x,0) = \int_{-\infty}^{\infty} g(\lambda)e^{\frac{i}{\hbar}\lambda_1 x} d\lambda \quad , \tag{3.6}$$

where $g(\lambda)$ is an arbitrary weight function (Fourier transform of $\psi(x,0)$). Then from (2.4), (3.4) and (3.6) we obtain the desired solution of equation (2.1)

$$\psi(x,t) = \int_{-\infty}^{\infty} g(\lambda)\psi_\lambda(x,t)d\lambda \quad , \tag{3.7}$$

(3.7) is a generalization of the formula (13) of Ref. [7] to the case of the time-dependent mass. If we now choose $g(\lambda) = \exp\left(i\lambda^3/3B^3\right)/2\pi$ and use the integral representation of the Airy function (A.12), then we find after integrating (3.7):

$$\begin{aligned} \psi(x,t) = B\exp&\left\{-\frac{i}{\hbar A_0}\left[C(t)x + \frac{1}{2A_0}\int_0^t \frac{C^2(t')}{M(t')}dt'\right]\right\}\times \\ \times\exp&\left\{\frac{iB^3}{2\hbar^2 A_0^3}\int_0^t \frac{dt'}{M(t')}\left[x + \frac{1}{A_0}\int_0^t \frac{C(t')}{M(t')}dt' - \frac{B^3}{6\hbar A_0^3}\left(\int_0^t \frac{dt'}{M(t')}\right)^2\right]\right\}\times \\ \times Ai&\left(\frac{B}{\hbar A_0}\left[x + \frac{1}{A_0}\int_0^t \frac{C(t')}{M(t')}dt' - \frac{B^3}{4\hbar A_0^3}\left(\int_0^t \frac{dt'}{M(t')}\right)^2\right]\right). \end{aligned} \tag{3.8}$$

When $M(t) = m$ one gets from the equation (3.8) the following formula [18]

$$\begin{aligned} \psi(x,t) = B\exp&\left\{-\frac{i}{\hbar A_0}\left[C(t)x + \frac{1}{2mA_0}\int_0^t C^2(t')dt'\right]\right\}\times \\ \times\exp&\left\{\frac{iB^3 S}{2\hbar^2 A_0^3}\left[x + \frac{1}{mA_0}\int_0^t C(t')dt' - \frac{B^3 S^2}{6\hbar A_0^3}\right]\right\}\times \\ \times Ai&\left(\frac{B}{\hbar A_0}\left[x + \frac{1}{mA_0}\int_0^t C(t')dt' - \frac{B^3 S^2}{4\hbar A_0^3}\right]\right), \end{aligned} \tag{3.9}$$

where $S = t/m$.

One can easily check that the functions (3.4) and (3.8) satisfy the Schrödinger equation (2.1).

4. MOMENTUM SPACE

We can solve the problem in the momentum *p*-space by evolution operator method. We write the Schrödinger equation (2.1) in the momentum space

$$i\hbar\partial_t\Phi(p,t) = \left[\frac{p^2}{2M(t)} - i\hbar F(t)\partial_p\right]\Phi(p,t) \ . \tag{4.1}$$

The evolution operator in the *p*-space has the simple form [15]: