REEXAMINATION A TIME-DEPENDENT HARMONIC OSCILLATOR

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The evolution operator method has been developed for the study of the time-dependent harmonic oscillator. The invariants and various class of states for the system under consideration are constructed by this method.

Keywords: Time-dependent harmonic oscillator, evolution operator, invariants, exact states.
PACS: 03.65.-w, 03.65.Ge, 03.65.Fd.

1. INTRODUCTION

It is well known that the time-dependent systems for which exact quantum mechanical solutions for the Schrödinger equation can be found are few in number. Both time-dependent harmonic oscillator [1-9] and the time-dependent linear potential [10-14] may be cited as examples. There have been different methods for finding the exact states of the time-dependent systems, such as Lewis-Riesenfeld (LR) invariant method [2], path integral method of Feynman [15], ordinary space-time transformations method [1,9,13] , evolution operator method [16] etc.

The most widely studied time-dependent system is the one-dimensional harmonic oscillator with time-dependent masses or both simultaneously. Besides the intrinsic mathematical interest, the time-dependent harmonic oscillator has many applications in various areas of physics, for example, in quantum optics, plasma physics, molecular physics and quantum chemistry. The harmonic oscillator undoubtedly plays a fundamental role in science.

First the problem of a one-dimensional quantum oscillator with time- dependent frequency moving under the action of a time-dependent force was exactly solved by Husimi [1], who had constructed for this problem Gaussian-type wave packets.

For the oscillator of constant mass and time-dependent frequency LR have introduced [2] an important quantum mechanical invariant I(t) and found the exact quantum states in terms of the invariant eigenstates. Recall that for a system describing by a time-dependent Schrödinger equation

$$\hat{S}(t)\Psi(t)=0, \qquad \hat{S}(t)=i\hbar\,\partial_t-H(t) \quad (1.1)$$

a Hermitian or non-Hermitian operator $I(t)$ is called an invariant if it commutes with the Schrödinger operator $\hat{S}(t)$, i.e.$[\hat{S}(t),I(t)]=0$, or, equivalently, satisfies the equation $i\hbar\,\partial_t I(t)+[I(t),H(t)]=0$. Lewis and Riesenfeld showed [2] that a solution $\Psi_\lambda(t)$ of the time-dependent Schrödinger equation (1.1) and an eigenfunction $\varphi_\lambda(t)$ of $I(t)$ corresponding to an eigenvalue λ $(I(t)\varphi_\lambda(t)=\lambda\varphi_\lambda(t)$ are connected by the relation

$$\Psi_\lambda(t)=e^{i\alpha_\lambda(t)}\varphi_\lambda(t), \quad (1.2)$$

where the time-dependent phase $\alpha_\lambda(t)$ is determined from the Schrödinger equation for $\Psi_\lambda(t)$.

Later the LR invariant method has been generalized to include time-dependent mass and driving force. As a result, exact wave functions have been obtained for harmonic oscillators with time-dependent frequency [3, 4], time-dependent mass and frequency [5-7] and driving force [17]. Two unitary relations between the systems of time-dependent harmonic oscillators were considered in Ref. [8]. The first relation is between the systems of time-dependent mass and of unit mass. The second relation is between those of the driven oscillator and undriven (see, also, [1]). Ciftja [9] found a solution of the harmonic oscillator with time-dependent mass and frequency by employing some simple space-time transformations.

The purpose of the present paper is to study the general problem of the harmonic oscillator with time-dependent mass and frequency moving under the action of a time-dependent force by using an evolution operator method. As is well known this method has been long time used to solve the problems in quantum mechanics and quantum field theory. In our study we have found that the evolution operator method is much simpler for deriving the quantum-mechanical quantities than other methods.

The evolution operator U(t) obeys Schrödinger equation (1.1) $\hat{S}(t)U(t)=0$ with the obvious initial condition U(0)=1. According to the principles of quantum mechanics all the information on the dynamics of quantum system is contained in the matrix elements of the evolution operator.

$$U(t)=T\exp\left(-\frac{i}{\hbar}\int_0^t H(t')dt'\right). \quad (1.3)$$

The evolution operator of the wave function $\Psi(t)$ (the solution of the Schrödinger equation (1.1)) is determined by the evolution operator, i.e.

$$\Psi(t)=U(t)\Psi(0), \quad (1.4)$$

where $\Psi(0)$ is initial wave function.

From the general representation (1.4) for the solutions of the Schrödinger equation (1.1) .It is evident

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that this equation has infinitely many solutions. Using different initial wave functions $\Psi(0)$, one can construct different wave functions $\Psi(t)$ at any later time $t > 0$

An evolution operator allows us also to construct the LR invariants for a given quantum system. The point is that any invariant for a quantum system can be expressed in terms of (through) two linearly independent simple invariants, such as $\hat{p}_0(t)$ and $\hat{x}_0(t)$ which are the initial momentum and coordinate operators, respectevly:

$$\hat{p}_0(t) = U(t)\hat{p}U^{-1}(t), \quad \hat{x}_o(t) = U(t)\hat{x}U^{-1}(t). \tag{1.5}$$

Therefore, any invariant $I(t)$ can be represented in the form $I(t) = U(t)I(0)U^{-1}(t)$, where $I(0) = G(\hat{p}, \hat{x})$ is any function of $\hat{p}$ and $\hat{x}$.

2. THE EVOLUTION OPERATOR

The Schrödinger equation for the harmonic oscillator with time-dependent mass $M(t)$ and frequency $\omega(t)$under the force $F(t)$ is

$$i\hbar\partial_t\Psi(t) = H(t)\Psi(t) \tag{2.1a}$$

with the time-dependent Hamiltonian

$$H(t) = -\frac{\hbar^2}{2M(t)}\partial_x^2 + \frac{1}{2}M(t)\omega^2(t)x^2 - F(t). \tag{2.1b}$$

It is convenient at first to reduce the problem (2.1) to the simpler case of vanishing $F(t)$. For this aim we perform an unitary transformation (compare with [1])

$$\Psi(x,t) = U_1(t)\Psi_1(x,t) \tag{2.2}$$

with the unitary operator

$$U_1(t) = \exp[-\xi(t)\partial_x]\exp\{\frac{i}{\hbar}[M(t)\dot{\xi}(t)x + \sigma(t)]\}, \tag{2.3}$$

where $\xi(t)$ satisfies the classical equation of motion

$$\frac{d}{dt}[M(t)\dot{\xi}(t)] + M(t)\omega^2(t)\xi(t) = F(t) \tag{2.4}$$

and $\sigma(t)$ is the classical action for the harmonic oscillator

$$\sigma(t) = \int_0^t [\frac{1}{2}M(t')\dot{\xi}^2(t') - \frac{1}{2}M(t')\omega^2(t')\xi^2(t') + F(t')\xi(t')dt'. \tag{2.5}$$

Note that $\xi(t)$ may, without loss of generality, be assumed to be that solution of equation (2.4), which initially vanishes together with its derivative $\dot{\xi}(t)$ [1], i.e.

$$\xi(0) = 0, \dot{\xi}(0) = 0. \tag{2.6}$$

From this it is evident that, the initial condition $U_1(0) = 1$ for $U_1(t)$ is holds. Moreover, it is clear that if $F(t) \equiv 0$, than $\xi(t) \equiv 0$.

As result if this fact the eq. (2.1a) has the following form:

$$i\hbar\partial_t\Psi_1(x,t) = H_1(t)\Psi_1(x,t), \tag{2.7a}$$

where

$$H_1(t) = -\frac{\hbar^2}{2M(t)}\partial_x^2 + \frac{1}{2}M(t)\omega^2(t)x^2. \tag{2.7b}$$

The wave function satisfies the initial condition $\Psi_1(x,0) = \Psi(x,0)$.

In the next step we take the wave function $\Psi_1(x,t)$ in the form

$$\Psi_1(x,t) = U_2(t)\Psi_2(x,t), \; U_2(t) = e^{i\alpha(t)x^2}, \tag{2.8}$$

where the time-dependent real function $\alpha(t)$ with the initial condition $\alpha(0) = 0$ is to be found later.

Substituting equation (2.8) into (2.7a) we obtain

$$i\hbar\partial_t\Psi_2(x,t) = \{-\frac{\hbar^2}{2M(t)}\partial_x^2 + [\hbar\,\dot{\alpha}(t) + \frac{2\hbar^2}{M(t)}\alpha^2(t) + \frac{1}{2}M(t)\omega^2(t)]x^2 - \frac{i\hbar^2}{M(t)}\alpha(t)(\partial_x x + x\partial_x)\}\Psi_2(x,t) \tag{2.9}$$

To simplify this equation we choose the auxiliary time-dependent function $\alpha(t)$ in such a way that the coefficient of x^2 vanishes. As a result we easily find that $\alpha(t)$ must satisfy the first-order nonlinear differential equation (Riccati equation)

$$\dot{\alpha}(t) + \frac{2\hbar}{M(t)}\alpha^2(t) = -\frac{1}{2\hbar}M(t)\omega^2(t). \quad (2.10)$$

This condition put on the auxiliary time-dependent function $\alpha(t)$ allows us to write equation (2.9) as

$$i\hbar\partial_t\Psi_2(x,t) = H_2(t)\Psi_2(x,t), \quad (2.11)$$

where the time-dependent Hamiltonian $H_2(t)$ is equal to

$$H_2(t) = -\frac{\hbar^2}{2M(t)}\partial_x^2 - \frac{2i\hbar^2}{M(t)}\alpha(t)\partial_x x + \frac{i\hbar^2}{M(t)}\alpha(t). \quad (2.12)$$

Introducing instead of $\alpha(t)$ a new function $\eta(t)$ by the relation

$$\alpha(t) = \frac{M(t)\dot{\eta}(t)}{2\hbar\,\eta(t)} \quad (2.13)$$

we transform the Riccati equation (2.12) to the form of the linear differential equation of the second-order:

$$\frac{d}{dt}[M(t)\dot{\eta}(t)] + M(t)\omega^2(t)\eta(t) = 0\,. \quad (2.14)$$

From the condition $\alpha(0) = 0$ follows that $\eta(0) \neq 0$ and $\dot{\eta}(0) = 0$.

Finally, we want to find the evolution operator $U_3(t)$ for the equation (2.11) with the Hamiltonian (2.12):

$$U_3(t) = T\exp\{-\frac{i}{\hbar}\int_0^t H_2(t')dt'\},\ \ U_3(0) = 1. \quad (2.15)$$

Note that a method to disentangle this type exponential operators into a product of the exponential operators was given in Ref. [18] (see, Appendix). Using this method, one can represent the evolution operator $U_3(t)$ (2.15) as follows:

$$U_3(t) = e^{\frac{1}{2}\delta(t)(\partial_x x + x\partial_x)} e^{iS(t)\partial_x^2}, \quad (2.16)$$

where

$$\delta(t) = -2\hbar\int_0^t \frac{\alpha(t')}{M(t')}dt', \qquad S(t) = \hbar\int_0^t \frac{e^{2\delta(t')}}{2M(t')}dt'. \quad (2.17)$$

It is clear that the evolution operator $U(t)$ for the equation (2.11) is just the product of the operators $U_1(t)$ (2.3), $U_2(t)$ (2.8) and $U_3(t)$ (2.16), i.e. $U = U_1U_2U_3$, or explicitly

$$U(t) = e^{-\frac{1}{2}\delta(t) + \frac{i}{\hbar}\sigma(t)} e^{-\xi(t)\partial_x} e^{\frac{i}{\hbar}M(t)\dot{\xi}(t)x} e^{i\alpha(t)x^2} e^{\delta(t)\partial_x x} e^{iS(t)\partial_x^2}. \quad (2.18)$$

This operator is unitary and satisfies the initial condition $U(0) = 1$. The solution of the Schrödinger equation (2.1) can be written now in the symbolic form

$$\Psi(x,t) = U(t)\Psi(x,0). \quad (2.19)$$

For simplicity in what follows we consider only the case $F(t) \equiv 0$. Then $U(t)$, instead of (2.18), will be givem by

$$U(t) = e^{-\frac{1}{2}\delta(t)} e^{i\alpha(t)x^2} e^{\delta(t)\partial_x x} e^{iS(t)\partial_x^2}. \tag{2.20}$$

As an example, we find an explicit form of the operator (2.20) for the particular case, when $M(t) = m$ and $\omega(t) = \omega_o$. In this case η is the solution of the equation $\ddot{\eta}(t) + \omega_o^2 \eta(t) = 0$, which must satisfy the conditions $\eta(0) = \eta_0$, $\dot{\eta}(0) = 0$. This solution is $\eta = \eta_0 \cos\omega_0(t)$. Then we easily obtain from (2.13), (2.16) and (2.17):

$$\alpha(t) = -\frac{m\omega_0}{2\hbar}\tan(\omega_0 t),$$

$$\delta(t) = -\ln(\cos\omega_0 t), \tag{2.21}$$

$$s(t) = \frac{\hbar}{2m\omega_0}\tan(\omega_0 t).$$

Substituting (2.21) into (2.20) one get the following expression for the evolution operator

$$U(t) = e^{-\frac{im\omega_0}{2\hbar}\tan(\omega_0 t)x^2} e^{\frac{i\hbar}{4m\omega_0}\sin(2\omega_0 t)\partial_x^2} e^{-\ln(\cos\omega_o t)(x\partial_x + 1/2)}, \tag{2.22}$$

which coincides with the well known formula (see, for example, [19]).

As is known, the time dependent Schrödinger equation, describing any quantum system has in finitely many solutions. Having found the evolution operator (2.20) and appropriately choosing the initial wave function in (2.19) one can construct any solution of the equation (2.1). The evolution operator method allows as to construct also LR invariants for the given system by the simple way.

3. INVARIANTS

First of all we find an explicit form of $\hat{p}_0$ and $\hat{x}_0$. Taking into account the explicit form (2.20) of the evolution operator we get

$$\hat{p}_0(t) = a_1(t)\hat{p} - M(t)\dot{a}_1(t)\hat{x}, \quad \hat{p}_0(0) = \hat{p},$$
$$\hat{x}_0(t) = a_2(t)\hat{p} - M(t)\dot{a}_2(t)\hat{x}, \quad \hat{x}_0(0) = \hat{x}, \tag{3.1}$$

where $a_1(t) = \exp[-\delta(t)]$, $a_2(t) = -2S(t)a_1(t)/\hbar$. We now define the time-dependent annihilation and creation operators

$$a(t) = \frac{1}{\sqrt{2}}\left(\lambda\hat{x}_0 + \frac{i\hat{p}_0}{\lambda\hbar}\right), \qquad a^+(t) = \frac{1}{\sqrt{2}}\left(\lambda\hat{x}_0 - \frac{i\hat{p}_0}{\lambda\hbar}\right), \tag{3.2}$$

where $\lambda = (m\omega_0/\hbar)^{-1/2}$, $m = M(0)$ and $\omega_0 = \omega(0)$. Next, let us rewrite the operators (3.2) as

$$a(t) = \frac{i}{\sqrt{2\hbar}}\left[\varepsilon(t)\hat{p} - M(t)\dot{\varepsilon}(t)x\right], \qquad a^+(t) = \frac{-i}{\sqrt{2\hbar}}\left[\varepsilon^*(t)\hat{p} - M(t)\dot{\varepsilon}^*(t)x\right] \tag{3.3}$$

with the function

$$\varepsilon(t) = (m\omega_0)^{-1/2}[1 + 2im\omega_0 S(t)/\hbar]a_1(t), \tag{3.4}$$

which satisfies the initial conditions $\varepsilon(0) = (m\omega_0)^{-1/2}$ and $\dot{\varepsilon}(0) = i(\omega_0/m)^{1/2}$. Using (2.10) one can show that the complex function $\varepsilon(t)$ obeys the equation (compare with Ref. [20], where the $M(t) = m$ case was considered)

$$\frac{d}{dt}\left[M(t)\dot{\varepsilon}(t)\right]+M(t)\omega^2(t)\varepsilon(t)=0 \tag{3.5}$$

and the relation $M\left(\dot{\varepsilon}\varepsilon^* - \dot{\varepsilon}^*\varepsilon\right)=2i$.

Let us put $\varepsilon(t)\equiv\rho(t)\exp\left[i\gamma(t)\right]$, where $\rho(t)=|\varepsilon(t)|$ and $\gamma(t)=\int_0^t dt'/M(t')\rho^2(t')$. Then from the equation (3.5) it follows that the function $\rho(t)$ satisfies an equation

$$\ddot{\rho}(t)+\frac{\dot{M}(t)}{M(t)}\dot{\rho}(t)+\omega^2(t)\rho(t)=\frac{1}{M^2(t)\rho^3(t)}\ . \tag{3.6}$$

We can express all quantities in terms of function $\rho(t)$. For example, we have

$$\delta(t)=-\ln\left(\rho\cos\gamma/\rho_0\right),$$

$$S(t)=\tan\gamma/2\lambda^2,$$

$$\alpha(t)=\left(\frac{M\dot{\rho}}{\rho}-\frac{\tan\gamma}{\rho^2}\right)/2\hbar,$$

$$\hat{p}_0(t)=\frac{1}{\rho_0}\left[\frac{\hat{x}}{\rho_0}\sin\gamma+\left(\rho\hat{p}-M\dot{\rho}\hat{x}\right)\cos\gamma\right], \tag{3.7a}$$

$$\hat{x}_0(t)=\rho_0\left[\frac{\hat{x}}{\rho}\cos\gamma-\left(\rho\hat{p}-M\dot{\rho}\hat{x}\right)\sin\gamma\right], \tag{3.7b}$$

$$a(t)=\frac{1}{\sqrt{2\hbar}}\left[\frac{\hat{x}}{\rho}+i\left(\rho\hat{p}-M\dot{\rho}\hat{x}\right)\right]e^{i\gamma},$$

$$a^+(t)=\frac{1}{\sqrt{2\hbar}}\left[\frac{\hat{x}}{\rho}-i\left(\rho\hat{p}-M\dot{\rho}\hat{x}\right)\right]e^{-i\gamma}, \tag{3.7c}$$

$$a_1(t)=\rho\cos\gamma/\rho_0,\qquad a_2(t)=-\rho_0\rho\sin\gamma,$$

where $\rho_0\equiv\rho(0)$. In the simple ease, when $M(t)=m$, $\omega(t)=\omega_0$,we have

$$\hat{p}_0=\hat{p}\cos\omega_0(t)+m\omega_0 x\sin\omega_0 t,\quad \hat{x}_0(t)=-\frac{\hat{p}}{m\omega_0}\sin\omega_0 t+x\cos\omega_0 t.$$

The operators $a(t)$ and $a^+(t)$, in deference of operators a и a^+ in [6], are the linear invariants. From them we construct the quadratic invariant

$$I_2(t)=a^+(t)a(t)+\hbar/2=\frac{1}{2}\left[\frac{x^2}{\rho^2}+\left(\rho\hat{p}-M\dot{\rho}x\right)^2\right]. \tag{3.8}$$

This formula is a generalization LR invariant [2] to the case of time-dependent mass [7] [21].

4. WAVE FUNCTIONS

From the general (symbolic) representation

$$\Psi(x,t)=e^{\frac{1}{2}\delta(t)}e^{i\alpha(t)x^2}e^{\delta(t)x\partial_x}e^{iS(t)\partial_x^2}\Psi(x,0) \tag{4.1}$$

for the solutions of the time-dependent Schrödinger equation (2.1) ($F(t)\equiv 0$) we can construct infinitely many wave functions. We will consider there type wave functions:

1) Plane wave type initial wave function

$$\Psi(x,t)=\frac{1}{\sqrt{2\pi\hbar}}\exp\left(ip_0x/\hbar\right). \tag{4.2}$$

Using the formula (A.3) at $\lambda=0$ we get the wave function

$$\Psi(x,t)=\frac{1}{\sqrt{2\pi\hbar}}\exp\left[\frac{1}{2}\delta(t)+\frac{ip_0x}{\hbar}e^{\delta(t)}+i\alpha(t)x^2-\frac{iS(t)}{\hbar^2}p_0^2\right]. \tag{4.3}$$

We express the wave function (4.3) in terms of $\rho(t)$:

$$\Psi(x,t)=\sqrt{\frac{\rho_0}{2\pi\hbar\rho\cos\gamma}}\exp\left\{\frac{i}{\hbar}\left[\frac{1}{2}\left(\frac{M\dot{\rho}}{\rho}-\frac{\tan\gamma}{\rho^2}\right)x^2-\frac{1}{2}\rho_0^2p_0^2\tan\gamma+\frac{\rho_0p_0x}{\rho\cos\gamma}\right]\right\}. \tag{4.4}$$

In the particular case, when $M(t)=m$, $\omega(t)=\omega_0$ we have

$$\Psi(x,t)=\frac{1}{\sqrt{2\pi\hbar\cos\omega_0t}}\exp\left[-\frac{i}{\hbar}\left(\frac{m\omega_0x^2}{2}+\frac{p_0^2}{2m\omega_0}-\frac{p_0x}{\sin\omega_0t}\right)\tan\omega_0t\right]. \tag{4.5}$$

2). Airy type initial wave function

$$\Psi(x,t)=Ai(Bx). \tag{4.6}$$

With the help of the formula (A.4) we find that in this case the wave function has a form

$$\Psi(x,t)=\exp\left[\frac{1}{2}\delta(t)+i\alpha(t)x^2\right]\exp\left[iS(t)B^3e^{\delta(t)}x-\frac{2}{3}iS^3(t)B^6\right]A\left(Be^{\delta(t)}x-S^2(t)B^4\right) \tag{4.7}$$

or in terms of $\rho(t)$ we have

$$\Psi(x,t)=\sqrt{\frac{\rho_0}{\rho\cos\gamma}}\exp\left\{-\frac{iB^6\tan^3\gamma}{12\lambda^6}+\frac{i\rho_0B^3\tan\gamma}{2\lambda^2\rho\cos\gamma}x+\frac{i}{2\hbar}\left(\frac{M\dot{\rho}}{\rho}-\frac{\tan\gamma}{\rho^2}\right)x^2\right\}\cdot$$
$$\cdot Ai\left(\frac{\rho_0B}{\rho\cos\gamma}x-\frac{B^4\tan^2\gamma}{4\lambda^4}\right) \tag{4.8}$$

If we take in (4.8) $M(t)=m$ and $\omega(t)=\omega_0$, then

$$\Psi(x,t)=\frac{1}{\sqrt{\cos\omega_0 t}}\exp\left\{-\frac{iB^6\tan^3\omega_0 t}{12\lambda^6}+\frac{iB^3\tan\omega_0 t}{2\lambda^2\cos\omega_0 t}x-\frac{\lambda^2\tan\omega_0 t}{2}x^2\right\}\cdot$$

$$\cdot Ai\left(\frac{Bx}{\cos\omega_0 t}-\frac{B^4\tan^2\omega_0 t}{4\lambda^4}\right) \tag{4.9}$$

3) Oscillator type initial wave function:

$$\Psi_n(x,0)=c_n e^{-\frac{\lambda^2x^2}{2}}H_n(\lambda x),\quad c_n=c_0/\sqrt{2^n n!}, \tag{4.10}$$

$$c_0=\left(\lambda^2/\pi\right)^{1/4}.$$

Using now (A.5) in (4.1) one obtains then the wave function corresponding to (4.10)

$$\Psi_n(x,t)=c_n\frac{e^{\frac{1}{2}\delta(t)}}{\sqrt{1+2i\lambda^2S(t)}}\left(\frac{1-2i\lambda^2S(t)}{1+2i\lambda^2S(t)}\right)^{\frac{n}{2}}e^{Ax^2}H_n\left(\frac{\lambda e^{\delta(t)}x}{\sqrt{1+4\lambda^4S^2(t)}}\right), \tag{4.11}$$

where

$$A=i\alpha(t)-\frac{\lambda^2e^{2\delta(t)}}{2\left(1+2i\lambda^2S(t)\right)}.$$

In terms of the function $\rho(t)$ the wave function (4.11) takes the form

$$\Psi_n(x.t)=c_n(\rho_0/\rho)^{\frac{1}{2}}e^{-i\left(n+\frac{1}{2}\right)\gamma}\exp\left[\frac{i}{2\hbar}\left(\frac{M\dot\rho}{\rho}+\frac{i}{\rho^2}\right)x^2\right]H_n\left(\frac{1}{\sqrt{\hbar}}\frac{x}{\rho}\right), \tag{4.12}$$

which agrees with the result of Ref. [7]. The functions (4.12) are the eigenfunctions of the invariant $I_2(t)$ (3.8) with the eigenvalues $\lambda_n=\left(n+\frac{1}{2}\right)\hbar$ and $a(t)\Psi_n(x,t)=\sqrt{n}\Psi_{n-1}(x.t)$, $a^+(t)\Psi_n(x,t)=\sqrt{n+1}\Psi_{n+1}(x,t)$.

In the particular case, when $M(t)=m$ and $\omega(t)=\omega_0$ from (4.12) it follows the well-known result:

$$\Psi_n(x,t)=c_ne^{-i\left(n+\frac{1}{2}\right)\omega_0(t)}e^{-\frac{\lambda^2x^2}{2}}H_n(\lambda x)=e^{-i\left(n+\frac{1}{2}\right)\omega_0 t}\Psi_n(x,0). \tag{4.13}$$

5. CONCLUSION

We have studied Schrödinger equation for a time-dependent harmonic oscillator with the help of the evolution operator method. Our analysis has shown that the key of solving the time-dependent Schrödinger equation is to find an evolution operator $U(t)$ of the system. This is explained by the following facts: 1) In this case, unlike LR invariant method, there is no further problem of finding the time-dependent phase; 2)The general representation for the wave function in terms of the evolution operator $\psi(t)=U(t)\psi(0)$ allows us to get any kinds of solutions of Schrödinger equation and with muchless efforts; 3) The evolution operator allows us to find not only solutions of the time-dependent Schrödinger equation, but also to construct all kinds of invariants.

APPENDIX

1. Note that operators $e^{\delta x\partial_x}$ and $e^{\alpha\partial_x^2}$ acts on functions $f(x)$ in following form:

$$e^{\delta x\partial_x} f(x) = f(e^{\delta} x), \tag{A.1}$$

$$e^{\alpha\partial_x^2} f(x) = \frac{1}{\sqrt{4\pi\alpha}} \int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{4\alpha}} f(z)dz = \sum_{-\infty}^{\infty} \frac{\alpha^n}{n!} f^{(2n)}(x). \tag{A.2}$$

Examples:

1) $$e^{\alpha\partial_x^2} e^{-\lambda x^2+\gamma x} = (1+4\alpha\lambda)^{-1/2} \exp\left[-\left(\lambda x^2 - \gamma x - \alpha\gamma^2\right)/(1+4\alpha\lambda)\right], \tag{A.3}$$

2) $$e^{\alpha\partial_x^2} e^{\gamma x} Ai(Bx) = e^{\varphi(x)} Ai\left(Bx + 2\alpha\gamma B + \alpha^2 B^4\right),$$

$$\varphi(x) = \left(\gamma + \alpha B^3\right)x + \alpha\gamma\left(\gamma + 2\alpha B^3\right) + \frac{2}{3}\alpha^3 B^6, \tag{A.4}$$

3) $$e^{\alpha\partial_x^2} e^{-\frac{1}{2}\lambda^2 x^2} H_n(\lambda x) = \left(1+2\alpha\lambda^2\right)^{-1/2} \left(\frac{1-2\alpha\lambda^2}{1+2\alpha\lambda^2}\right)^{n/2} \exp\left[-\frac{\lambda^2 x^2}{2\left(1+2\alpha\lambda^2\right)}\right] H_n\left(\frac{\lambda x}{\sqrt{1-2\alpha^2\lambda^4}}\right), \tag{A.5}$$

where $Ai(x)$ is the Airy function and $H_n(\lambda x)$ is Hermite polynomial. To derive the formula (A.4) we have used the integral [22]

$$\int_{-\infty}^{\infty} e^{-p(x-y)^2} H_n(cx)dx = \sqrt{\frac{\pi}{p}} \left(\frac{p-c^2}{p}\right)^{n/2} H_n\left(cy\sqrt{\frac{p}{p-c^2}}\right), \ y, \ \mathrm{Re}\, p > 0. \tag{A.6}$$

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[22] If a printing error in eq. (4) Ref. [7] is corrected, i. e. replacing $\left(\frac{q}{\rho}\right)^{\frac{1}{2}}$ with $\left(\frac{q}{\rho}\right)^{2}$. We can easily find that eq. (4) of Ref. [7] is identical to eq. (3.8) in the present paper.

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Received: 19.09.2016