# THE CONFINED HARMONIC OSCILLATOR AS AN EXPLICIT SOLUTION OF THE POSITION-DEPENDENT EFFECTIVE MASS SCHRODINGER EQUATION WITH MORROW-BROWNSTEIN HAMILTONIAN

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Exactly-solvable confined model of the non-relativistic quantum harmonic oscillator with Morrow-Brownstein kinetic energy operator $H_0 = M^{\alpha}(x)\hat{p}_x M^{\beta}(x)\hat{p}_x M^{\alpha}(x)/2$ (with $2\alpha + \beta = -1$) is proposed. Corresponding position-dependent effective mass Schrödinger equation in the canonical approach is solved in position representation. Explicit expressions of both wavefunctions of the stationary states and discrete energy spectrum have been found. It is shown that the energy spectrum has non-equidistant form and depends on both confinement parameter $a$ and Morrow-Brownstein parameter $\alpha$. Wavefunctions of the stationary states in position representation are expressed in terms of the Gegenbauer polynomials. At limit $a \to \infty$, both energy spectrum and wavefunctions recover well-known equidistant energy spectrum and wavefunctions of the stationary non-relativistic harmonic oscillator expressed by Hermite polynomials. Position dependence of the effective mass also disappears under the same limit.

**Keywordas:** Morrow-Brownstein kinetic energy operator, confined harmonic oscillator model, exact solution, Gegenbauer polynomials, non-equidistant energy spectrum, position-dependent effective mass

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## 1. INTRODUCTION

One-dimensional harmonic oscillator is one of the attracting problems of the quantum theory that has enormous applications in the different scientific branches of the modern physics and technologies [1]. Exact solubility of this problem within different approaches can be considered as its main attractive property. Explicit solution of its stationary Schrödinger equation within the canonical approach in the position or momentum representation is best known case with wide applications in physics and mathematics. Here, obtaining explicit expressions of the wavefunctions of the stationary states and discrete energy spectrum formally means exact solution of the problem under study. Stationary Schrödinger equation within the canonical approach in the position representation leads to the second order differential equation, whose eigenfunctions vanishing at infinity are wavefunctions of the one-dimensional quantum harmonic oscillator expressed in terms of the Hermite polynomials and eigenvalues are discrete energy spectrum with equidistant energy levels spacing [2]. This is not only the explicit quantum mechanical realization of the one-dimensional harmonic oscillator problem with wavefunctions vanishing at infinity. Explicit solved model of the one-dimensional non-relativistic harmonic oscillator is realized within non-canonical approach, whose wavefunctions of the stationary states are expressed through the generalized Laguerre polynomials [3]. Explicit realizations of the quantum harmonic oscillator with wavefunctions vanishing at infinity within the different relativistic formalisms also exist [4-6]. Another approach for exact solution of the quantum harmonic oscillator problem with wavefunctions vanishing at infinity is study of the problem in the discrete position of momentum space. Usually, wavefunctions of such models are expressed in terms of the finite- or infinite-discrete orthogonal polynomials [7-14]. One need to note another interesting hybrid model that is realized as quantum harmonic oscillator with wavefunctions vanishing at infinity and its wavefunctions are expressed by Meixner and generalized Laguerre polynomials depending on the discreteness or continuous nature of the configuration representation [15]. However, quantum-mechanical explicit solution of the harmonic oscillator problem in the infinite-continuous configuration space is still open. It is obvious that the wavefunctions of the stationary states of the explicitly solved quantum harmonic oscillator should differ from above listed models with property of vanish at finite region. Some attempts have been done to solve this oscillator problem, but all these attempts lead to approximate expressions both wavefunctions and energy spectrum [16-23]. Taking into account the role that such a harmonic oscillator model can play in the development of different branches of physics, especially, nanotechnologies and low dimensional physics, we suppose explicit quantum mechanical solution of the harmonic oscillator confined in finite region within the framework of the position-dependent effective mass formalism. Significance of the position-dependent effective mass formalism mainly is its appearance in quantum-mechanical solutions due to experimental data coming from solid state physics and necessity of its correct interpretation within quantum theory laws. For the first time, [24] used position-dependent band structure formalism in order to explain seminal experiment on the tunneling into superconductors [25, 26]. Then, this formalism was developed into the assumption that position-dependent band structure formalism should be simulated by position-dependent effective mass $M(x)$Based on this assumption, the non-relativistic Hermitian kinetic energy operator with effective mass varying with position was introduced in [27]. It is now known as BenDaniel-Duke kinetic energy operator:

$$\hat{H}_O^{BD} = \frac{1}{2}\hat{p}_x \frac{1}{M(x)}\hat{p}_x \quad (1)$$

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Later, one-dimensional model of an abrupt heterojunction with the same lattice constant throughout the structure was considered theoretically with assumption that all primitive lattice cells on two opposite sides of the hetero-interface differ from each other and another kind of non-relativistic kinetic energy operator with position-dependent effective mass was introduced as a result of the study of the problem of the connection rules for effective-mass wave functions across an abrupt heterojunction between two different semiconductors [28]. It is known as Zhu-Kroemer kinetic energy operator:

$$\hat{H}_O^{ZK} = -\frac{1}{\sqrt{M(x)}}\hat{p}_x^2\frac{1}{M(x)} \quad (2)$$

Further, Zhu-Kroemer kinetic energy operator formalism was supported as a model of an abrupt heterojunction between two different semiconductors via more general investigation in [29, 30]. It resulted introduction of the more generalized non-relativistic Hermitian operator with position-dependent effective mass that is known at present as Morrow-Brownstein kinetic energy operator:

$$\hat{H}_O^{MB} = \frac{1}{2}M^{\alpha}(x)\hat{p}_x M^{\beta}(x)\hat{p}_x M^{\alpha}(x), \ \ 2\alpha + \beta = -1 \quad (3)$$

It is obvious that BenDaniel-Duke and Zhu-Kroemer kinetic energy operators with $\alpha = 0$ ($\beta = -1$) and $\alpha = -\frac{1}{2}$ ($\beta = 0$)are special cases of Morrow-Brownstein kinetic energy operator. Schrödinger equation with position-dependent effective mass involving all three listed above kinetic energy operators is studied thoroughly in a lot of published works. For example, exact solutions of the BenDaniel-Duke Schrödinger equation with different potentials are obtained for position-dependent effective mass of exponential behavior by using coordinate transformation method in [31]. Also, [32] generalized harmonic oscillator model through explicit solution of the corresponding Morrow-Brownstein Schrödinger equation with position-dependent effective mass and obtains explicit expressions of the wavefunctions of the stationary states and energy spectrum. A one-dimensional harmonic oscillator with position-dependent effective mass is studied in [33] and corresponding BenDaniel-Duke Schrödinger equation is solved explicitly in terms of modified Hermite polynomials. Position-dependent effective mass formalism is not only powerful tool in the solution of the Schrödinger equation. For example, the Duffin–Kemmer–Petiau equation with position-dependent mass for relativistic spin-1 particles under Coulomb interaction as well as in the presence of Kratzer-type potential is also studied analytically leading to obtaining asymptotical solutions in terms of the eigenvalues and eigenvectors [34, 35].

Present paper is structured as follows: in Section 2, the basic known information about the one-dimensional quantum harmonic oscillator is presented. This information covers explicit expressions of the wavefunctions of the stationary states and energy spectrum obtained via solution of the non-relativistic Schrödinger equation within canonical approach. Section 3 is devoted to the solution of the Morrow-Brownstein Schrödinger equation with position-dependent effective mass for the confined quantum harmonic oscillator model. Explicit expressions of the both wavefunctions of the stationary states in terms of the Gegenbauer polynomials and non-equidistant energy spectrum are presented in this section as well as it is shown that obtained wavefunctions possess correct orthogonality relations. Brief discussion of the obtained results is given in Section 4. It is shown that both wavefunctions and energy spectrum correctly recover their unconfined expressions under certain limit.

## 2. NON-RELATIVISTIC QUANTUM HARMONIC OSCILLATOR UNDER THE CANONICAL APPROACH: WAVEFUNCTIONS IN TERMS OF HERMITE POLYNOMIALS AND EQUIDISTANT ENERGY SPECTRUM

Solution of the non-relativistic quantum harmonic oscillator is well known and it is obtained from the following stationary Schrödinger equation in position representation

$$\left[\frac{\hat{p}_x^2}{2m} + V(x)\right]\psi(x) = E\psi(x) \quad (4)$$

In order to solve this equation, first of all it is necessary to define non-relativistic harmonic oscillator potential that has the following form:

$$V(x) = \frac{m\omega^2 x^2}{2} \quad (5)$$

Here $m$ and $\omega$ are position-independent mass and angular frequency of the non-relativistic quantum harmonic oscillator. Then, it is necessary to define if this equation is going to be solved under canonical or non-canonical approach. Taking into account that we are interested in the solution under the canonical approach, then definition of the one-dimensional momentum operator under this approach is as follows

$$\hat{p}_x = -i\hbar\frac{d}{dx}, \quad (6)$$

whose substitution to Schrödinger equation (4) means that it is written in the canonical approach. Finally, one needs to look for explicit non-relativistic harmonic oscillator solutions with vanishing wavefunctions at infinity. Then, taking into account this statement and both definitions (5) and (6) in eq. (4) one easily obtains the following well-known second order differential equation:

$$\frac{d^2\psi}{d^2x} + \frac{2m}{\hbar^2}\left(E - \frac{m\omega^2 x^2}{2}\right)\psi = 0 \quad (7)$$

Explicit solutions allow to obtain eigenfunctions of this equation, which are wavefunctions of the stationary states in the position representation

$$\psi \equiv \psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \tag{8}$$

and eigenvalues of this equation in terms of discrete equidistant energy spectrum

$$E \equiv E = \hbar\omega\left(n+\frac{1}{2}\right),\ n = 0,1,2,\ldots, \tag{9}$$

where, $H_n(x)$ are Hermite polynomials defined through ${}_2F_0$ hypergeometric functions as follows [36]:

$$H_n(x) = (2x)^n\ {}_2F_0\left(-\frac{n}{2}.-(n-1)/2; -\frac{1}{x^2}\right) \tag{10}$$

Hermite polynomials satisfy the following orthogonality relation:

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)dx = 2^n n!\,\delta_{mn}, \tag{11}$$

therefore, normalized wavefunctions (8) based on (11) also satisfy similar orthogonality relation:

$$\int_{-\infty}^{\infty} \psi_m^*(x)\,\psi_n(x)dx = \delta_{mn} \tag{12}$$

In next section, we are going to generalize these expressions to confined Morrow-Brownstein quantum oscillator case.

## 3. CONFINED MORROW-BROWNSTEIN QUANTUM HARMONIC OSCILLATOR: WAVEFUNCTIONS IN TERMS OF GEGENBAUER POLYNOMIALS AND NON-EQUIDISTANT ENERGY SPECTRUM

We start from the stationary Schrödinger equation with position-dependent effective mass $M(x)$ and introduce confined harmonic oscillator potential as follows:

$$V(x) = \begin{cases} \frac{M(x)\omega^2x^2}{2}, & -a < x < a, \\ \infty, & x = \pm a. \end{cases} \tag{13}$$

Taking into account Morrow-Brownstein kinetic energy operator (13), it is clear that we need to solve the stationary Schrödinger equation with the following Hamiltonian:

$$H^{MB} = \frac{1}{2}M^{\alpha}(x)\hat{p}_x M^{\beta}(x)\hat{p}_x M^{\alpha}(x) + \frac{m\omega^2x^2}{2} \tag{14}$$

We are going to solve the stationary Schrödinger equation within canonical approach, therefore, substitution of the momentum operator $\hat{p}_x$ (6) to (14) and simple straightforward computations lead to the following expression of Hamiltonian $H^{MB}$:

$$H^{MB} = -\frac{\hbar^2}{2M}\left[\frac{d^2}{dx^2} - \frac{M'}{M}\frac{d}{dx} + \alpha\frac{M''}{M} - \alpha(\alpha+2)\left(\frac{M'}{M}\right)^2\right] + \frac{M\omega^2x^2}{2}, \tag{15}$$

with $M \equiv M(x)$.

It is clear that defining explicitly position-dependent effective mass $M \equiv M(x)$, one can try to solve explicitly the Schrödinger equation with Hamiltonian (15). We require that position-dependent effective mass $M(x)$ has to satisfy the following conditions:

- $M(x)$ should recover correct constant mass $m$ at origin of position $x = 0$ and then under the limit $a \to \infty$;
- Confinement effect at values of position $x = \pm a$ should be achieved via analytical definition of the position-dependent effective mass $M(x)$;
- Stationary Schrödinger equation for the Hamiltonian $H^{MB}$ (15) should be explicitly solvable and then its solutions should recover non-relativistic Hermite oscillator solutions under the limit $a \to \infty$;.

Based on listed above conditions, we define the following analytical expression for the position-dependent effective mass $M(x)$:

$$M \equiv M(x) = \frac{a^2 m}{a^2 - x^2} \tag{16}$$

It is clear that first condition is satisfied easily, because, $M(0) = m$ as well as

$$\lim_{a \to \infty} \frac{a^2 m}{a^2 - x^2} = m \tag{17}$$

Concerning second condition, one observes that harmonic oscillator potential (13) with position-dependent effective mass $M(x)$ (16) possesses the correct infinite high walls boundary conditions

$$V(-a) = V(a) = \infty \tag{18}$$

Then, we need only to prove the satisfaction of the third condition. Therefore, we have to solve explicitly stationary Schrödinger equation for the Hamiltonian $H^{MB}$ (15). Simple computations show that

$$\frac{M'}{M} = \frac{2x}{a^2 - x^2},$$
$$\frac{M''}{M} = \frac{2x}{a^2 - x^2} + \frac{8x^2}{(a^2 - x^2)^2}$$

Taking them into account in Hamiltonian $H^{MB}$ (15) we have

$$H^{MB} = -\frac{\hbar^2}{2M}\left[\frac{d^2}{dx^2} - \frac{2x}{a^2 - x^2}\frac{d}{dx} + \frac{2\alpha}{a^2 - x^2} - \frac{4\alpha^2 x^2}{(a^2 - x^2)^2}\right] + \frac{M\omega^2 x^2}{2}, \tag{19}$$

Corresponding Schrödinger equation becomes the following second order differential equation:

$$\left[\frac{d^2}{dx^2} - \frac{2x}{a^2 - x^2}\frac{d}{dx} + \frac{2\alpha}{a^2 - x^2} - \frac{4\alpha^2 x^2}{(a^2 - x^2)^2}\right]\psi + \left(\frac{c_0}{a^2 - x^2} - \frac{c_2}{(a^2 - x^2)^2}\right)\psi = 0, \tag{20}$$

with $c_0 = \frac{2ma^2 E}{\hbar^2}$ and $c_2 = \frac{m^2\omega^2 a^4}{\hbar^2}$.

Now, introduction of the dimensionless variable $\xi = \frac{x}{a}$ with $\frac{d}{dx} = \frac{1}{a}\frac{d}{d\xi}$ and $\frac{d^2}{dx^2} = \frac{1}{a^2}\frac{d^2}{d\xi^2}$ leads to the following second order differential equation:

$$\psi'' + \frac{\tilde{\tau}}{\sigma}\psi' + \frac{\tilde{\sigma}}{\sigma^2}\psi = 0 \tag{21}$$

Here

$$\tilde{\tau} = -2\xi, \tag{22}$$

$$\sigma = 1 - \xi^2, \tag{23}$$

and

$$\tilde{\sigma} = (2\alpha + c_0) - (4\alpha^2 + 2\alpha + c_0 + c_2)\xi^2 \tag{24}$$

We apply Nikiforov-Uvarov method for solution of eq.(21). This method is applicable to differential equations with $\sigma$ and $\tilde{\sigma}$ being polynomials at most of second degree and $\tilde{\tau}$ being a polynomial at most of first degree [37]. We look for solution of eq.(21) as follows:

$$\psi = \varphi(\xi) y \tag{25}$$

Here, $\varphi(\xi)$ is defined by the following manner:

$$\varphi(\xi) = e^{\int \frac{\pi(\xi)}{\sigma(\xi)} d\xi}, \tag{26}$$

with $\pi(\xi)$ also being a polynomial at most of first degree.

Simple computations with substitution of (25) in eq.(21) lead to the following differential equation for $y$:

$$y'' + \frac{\tau}{\sigma}y' + \frac{\bar{\sigma}}{\sigma^2}y = 0 \tag{27}$$

Here

$$\tau = 2\pi + \tilde{\tau}, \tag{28}$$

And

$$\bar{\sigma} = \tilde{\sigma} + \pi^2 + \pi(\tilde{\sigma} - \sigma') + \pi'\sigma \tag{29}$$

Already, the function $\pi(\xi)$ has been defined as a polynomial at most of first degree. We are going to choose the coefficients of $\pi(\xi)$ in $\bar{\sigma}$ (29) will be divisible by $\sigma$ as follows:

$$\bar{\sigma} = \lambda\sigma \tag{30}$$

Then, we have to obtain explicit expression of $\pi(\xi)$ by solving the following quadratic equation:

$$\pi^2 + \pi(\tilde{\sigma} - \sigma') + \tilde{\sigma} - \mu\sigma = 0, \tag{31}$$

with definition $\mu = \lambda - \pi'$. Taking into account that $\tilde{\tau} - \sigma' = 0$, quadratic equation (31) will be simplified as follows:

$$\pi^2 + \tilde{\sigma} - \mu\sigma = 0 \tag{32}$$

Its solution in general can be written as

$$\pi = \varepsilon\sqrt{(\mu - 2\alpha - c_0) + (4\alpha^2 + 2\alpha + c_0 + c_2 - \mu)\xi^2}, \varepsilon = \pm 1$$

Two cases (four solutions) satisfy the condition for $\pi(\xi)$ being a polynomial at most of first degree:

$$\pi(\xi) = \begin{cases} \varepsilon q\xi, & \mu = 2\alpha + c_0 \\ \varepsilon q, & \mu = 4\alpha^2 + 2\alpha + c_0 + c_2 \end{cases} \quad (33)$$

where, $q = \sqrt{(4\alpha^2 + c_2)}$. Finding explicit expressions of functions $\pi(\xi)$ and $\sigma(\xi)$ now one can also obtain explicit expression of $\varphi(\xi)$ from (26). Through simple computations we obtain that $\varphi(\xi) = (1 - \xi^2)^{-\frac{1}{2}\varepsilon q}$ for case $\mu = 2\alpha + c_0$ and $\varphi(\xi) = (\frac{1+\xi}{1-\xi})^{\frac{1}{2}\varepsilon q}$ for case $\mu = 4\alpha^2 + 2\alpha + c_0 + c_2$.

Finiteness property of the wavefunction at points $\xi + \pm 1$ (or $x = \pm a$) (i.e., $\lim_{\xi \to \pm 1} \varphi(\xi) = 0$ should be satisfied at all) requires that the case $\mu = 2\alpha + c_0$ should be chosen and the condition $\varepsilon = -1$ should be satisfied for $\varphi(\xi)$. These consequences lead to the following final expressions of $\pi(\xi)$ and $\varphi(\xi)$:

$$\pi = -q\xi = -\sqrt{4\alpha^2 + c_2}\,\xi,$$

$$\varphi(\xi) = (1 - \xi^2)^{\frac{q}{2}}$$

Now, one can easily compute multiplier $\lambda$ introduced in (30):

$$\lambda = 2\alpha + c_0 - q.$$

Eq.(27) via substitution of (30) will have the following simpler form:

$$\sigma y'' + \tau y' + \lambda y = 0.$$

Taking into account that $\tau$ also can be easily computed from (28) and it is equal to

$$\tau = -2(q + 1)\xi,$$

then, we need to look for the polynomial solution of the equation

$$(1 - \xi^2)y'' - 2(q + 1)\xi y' + (2\alpha + c_0 - q)y = 0 \quad (34)$$

Comparing eq.(34) with the following second order differential equation for the Gegenbauer polynomials [36]

$$(1 - x^2)\overline{y''} - (2\bar{\lambda} + 1)x\overline{y'} + n(n + 2\bar{\lambda})\bar{y} = 0, \bar{y} = C_n^{\bar{\lambda}}(x),$$

we obtain the following non-equidistant energy spectrum

$$E \equiv E_n^{MB} = \hbar\sqrt{\omega^2 + \frac{4\alpha^2\hbar^2}{m^2a^4}}\left(n + \frac{1}{2}\right) + \frac{\hbar^2}{2ma^2}(n^2 + n - 2\alpha), \quad (35)$$

and wavefunctions of the stationary states

$$\psi \equiv \psi_n^{MB}(x) = C_n^{MB}\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}\sqrt{\frac{m^2\omega^2a^4}{\hbar^2} + 4\alpha^2}} C_n^{\sqrt{\frac{m^2\omega^2a^4}{\hbar^2} + 4\alpha^2} + \frac{1}{2}}\left(\frac{x}{a}\right), \quad (36)$$

Where $C_n^{\bar{\lambda}}(x)$ are Gegenbauer polynomials defined in terms of the ${}_2F_1$ hypergeometric functions as follows:

$$C_n^{\bar{\lambda}}(x) = \frac{(2\bar{\lambda})_n}{n!}\, {}_2F_1\left(\begin{matrix}-n, n+2\bar{\lambda} \\ \bar{\lambda}+1/2\end{matrix}; \frac{1-x}{2}\right), \bar{\lambda} \neq 0 \quad (37)$$

and the normalization factor $C_n^{MB}$ being equal to

$$C_n^{MB} = 2^{\sqrt{\frac{m^2\omega^2a^4}{\hbar^2} + 4\alpha^2}}\Gamma\left(\sqrt{\frac{m^2\omega^2a^4}{\hbar^2} + 4\alpha^2} + \frac{1}{2}\right)\sqrt{\frac{\left(n + \sqrt{\frac{m^2\omega^2a^4}{\hbar^2} + 4\alpha^2} + \frac{1}{2}\right)n!}{\pi a\Gamma(n + 2\sqrt{\frac{m^2\omega^2a^4}{\hbar^2} + 4\alpha^2} + 1)}} \quad (38)$$

is defined from the orthogonality relation for Gegenbauer polynomials $C_n^{\bar{\lambda}}(x)$ of the following form

$$\int_{-1}^{1}(1-x^2)^{\bar{\lambda}-\frac{1}{2}}C_m^{\bar{\lambda}}(x)C_n^{\bar{\lambda}}(x)dx=\frac{\pi\Gamma(n+2\bar{\lambda})2^{1-2\bar{\lambda}}}{\{\Gamma(\bar{\lambda})\}^2(n+\bar{\lambda})n!}\delta_{nm},$$

that is valid within preliminary conditions $\bar{\lambda}>-\frac{1}{2}$ and $\bar{\lambda}=0$. Therefore, wavefunctions of the stationary states in the position representation (36) are also orthogonal in the finite range $-a<x<a$:

$$\int_{-a}^{a}[\psi_m^{MB}(x)]^*\,\psi_n^{MB}(x)=\delta_{nm} \qquad (40)$$

As well as explicit expressions of the energy spectrum (35) and the wavefunctions (36) are obtained, this proves exactly solubility of the quantum system under consideration within that we already defined position-dependent effective mass $M(x)$. We will discuss their behaviour as well as correct limit conditions under the limit $a\to\infty$ below.

## 4. DISCUSSIONS AND CONCLUSIONS

Taking into account that both energy spectrum and wavefunctions of the stationary states of the confined oscillator problem under study are known explicitly, one can discuss their different properties and special case behaviour. First of all, First of all, obtained energy spectrum (35) is not equidistant. This property is its main difference from the energy spectrum of the Hermite oscillator. We note in introduction that Morrow-Brownstein kinetic energy operator depends from the parameter $\alpha$ that is bounded with another parameter $\beta$ via the restriction $2\alpha+\beta=-1$. This is only the restriction for $\alpha$ parameter, therefore, its appearance under square root in first term of the expression of the energy spectrum (35) drastically changes its behaviour. We briefly note that for imaginary values of $\alpha$ square root $\sqrt{\omega^2+\frac{4\alpha^2\hbar^2}{m^2a^4}}$ can be also imaginary and this case with complex energy spectrum would have further attractive applications. Another important property of the non-equidistant energy spectrum of the Morrow-Brownstein model is that there is restriction on its positivity that mathematically can be written as

$$2qe_n>e_n^2-2\alpha-\frac{1}{4},$$

with $e_n=E_n/\hbar\omega$.

Already (17) proves correct limit of position-dependent mass to homogeneous one at $a\to\infty$. Similar limit relation exists for energy spectrum (35) that is the following:

$$\lim_{a\to\infty}E_n^{MB}=\hbar\omega\left(n+\frac{1}{2}\right)=E_n \qquad (41)$$

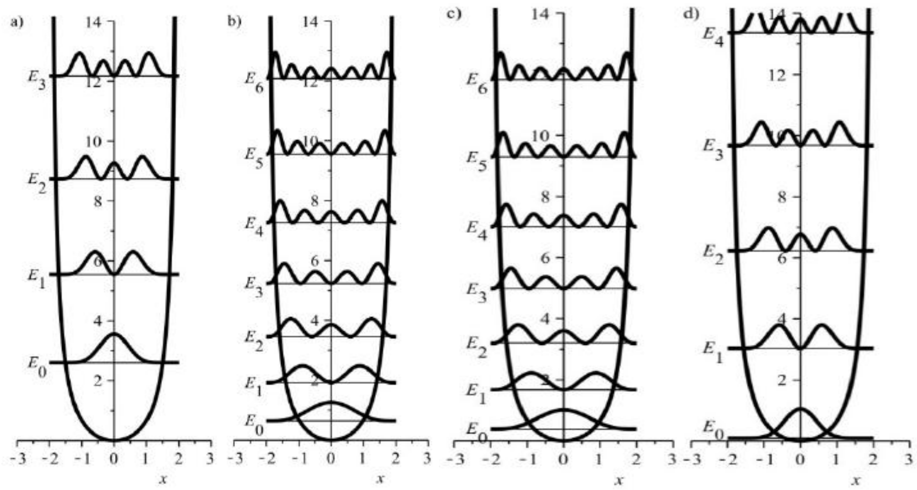

*Fig. 1.* Confined quantum harmonic oscillator potential (13) and behaviour of the corresponding non-equidistant energy levels (35) at value of the confinement parameter $a=2$ and probability densities $|\psi_n^{MB}(x)|^2$ of the ground and a) 3 excited states for value of the parameter $\alpha=-5$; b) 6 excited states for value of the parameter $\alpha=-0.5$; c) 6 excited states for value of the parameter $\alpha=-0.5$; d) 4 excited states for value of the parameter $\alpha=5$ ($m=\omega=\hbar=1$).

The limit between the confined and Hermite oscillators wavefunctions (36) and (8) is satisfied, too. Correctness of this limit is based on the existence of the following limit relation between Gegenbauer polynomials $C_n^{\bar{\lambda}}(x)$ and Hermite polynomials $H_n(x)$ [36]:

$$\lim_{a\to\infty}\bar{\lambda}^{-\frac{1}{2}n}C_n^{(\bar{\lambda}+\frac{1}{2})}\left(\frac{x}{\sqrt{\bar{\lambda}}}\right)=\frac{H_n(x)}{n!} \qquad (42)$$

Thanks to this limit relation, wavefunctions of the stationary states of the Morrow-Brownstein confined quantum harmonic oscillator with a position-dependent effective mass $\psi_n^{MB}(x)$ (36) reduce to wavefunctions of the stationary states of the Hermite oscillator potential $\psi_n(x)$ (8) under the following limit relation:

$$\lim_{a\to\infty} \psi_n^{MB}(x) = \psi_n(x) \qquad (43)$$

Following asymptotics and limit relations can be useful during derivation of (43):

$$\Gamma_{|z|\to\infty} \simeq \sqrt{\frac{2\pi}{z}}\, e^{zlnz - z},$$

$$\Gamma\left(\sqrt{\bar{\lambda}^2+4\alpha^2}+1/2\right)_{\bar{\lambda}\to\infty} \simeq \sqrt{2\pi} e^{\bar{\lambda}ln\bar{\lambda}-\bar{\lambda}}, \bar{\lambda} = \frac{m\omega a^2}{\hbar},$$

$$\Gamma\left(n + 2\sqrt{\bar{\lambda}^2 + 4\alpha^2} + 1\right)_{\bar{\lambda}\to\infty} \simeq 2^n\sqrt{4\pi\bar{\lambda}} e^{(2\bar{\lambda}+n)ln\bar{\lambda} - 2\bar{\lambda} + 2\bar{\lambda}ln2},$$

$$\lim_{\bar{\lambda}\to\infty} \bar{\lambda}^{\frac{n}{2}} c_n^{MB} = \tilde{c}_0 \sqrt{\frac{n!}{2^n}}, \tilde{c}_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}},$$

$$\lim_{\bar{\lambda}\to\infty} \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}\sqrt{\bar{\lambda}^2+4\alpha^2}} = e^{-\frac{m\omega x^2}{2\hbar}}$$

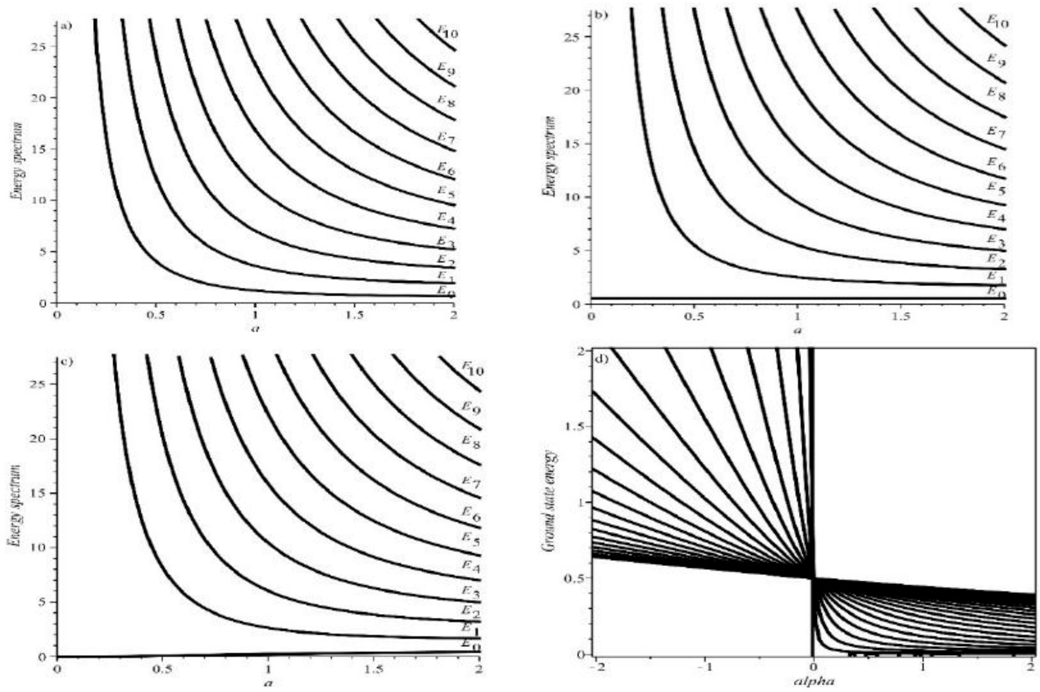


*Fig. 2.* Dependence of the non-equidistant energy levels (35) from the confinement parameter $a$ for the ground and 10 excited states ($m = \omega = \hbar = 1$): a) $\alpha = -\frac{1}{2}$; b) $\alpha = 0$; c) $\alpha = \frac{1}{2}$; and d) dependence of ground state energy level from the $\alpha$ parameter for values of $a = 0.4$

In fig.1, we depicted confined quantum harmonic oscillator potential (13) and behaviour of the corresponding non-equidistant energy levels (35) at value of the confinement parameter $a = 2$ and probability densities $|\psi_n^{MB}(x)|^2$ of the ground and a) 3 excited states for value of the parameter $\alpha = -5$; b) 6 excited states for value of the parameter $\alpha = -0.5$; c) 6 excited states for value of the parameter $\alpha = 0.5$; d) 4 excited states for value of the parameter $\alpha = 5$ ($m = \omega = \hbar = 1$). Dependence of the non-equidistant energy levels (35) from the confinement parameter $a$ for the ground and 10 excited states: a) $\alpha = -\frac{1}{2}$; b) $\alpha = 0$; c) $\alpha = \frac{1}{2}$; and d) dependence of ground state energy level from the $\alpha$ parameter for values of $a = 0.4$ are presented in fig.2 (all plots are depicted in $m = \omega = \hbar = 1$ system). Depicting dependence of the non-equidistant energy levels from the confinement parameter $a$ we observed that behaviour of the ground state for different values of $\alpha$ is completely different. Therefore, we decided to make a plot of the dependence of ground state energy level from parameter $\alpha$ for different values of the confinement parameter $a$. We observe that for some positive values of parameter $\alpha$, ground state energy can reduce to zero. This happens due to existence of parameter $\alpha$ in both terms of the energy spectrum expression. In fig.1 we also observe how infinite high walls appear for harmonic oscillator depending on value of $a$ and of course, we provide probability densities $|\psi_n^{MB}(x)|^2$ corresponding energy levels aiming to exhibit their behaviour within confinement effect. Due to that, we fix confinement parameter $a$ in our plots at value $a = 2$, infinite walls also appear at values of position $x = \pm 2$.

One can extend this discussion, however, important point here is demonstration of exactly solubility of the quantum harmonic oscillator with Morrow-Brownstein kinetic energy operator under the confinement effect and obtaining explicit expressions of the wavefunctions of stationary states and energy spectrum through solution of the corresponding Schrödinger equation. Method applied here can be generalized further for one-dimensional relativistic confined oscillator systems, which also can exhibit surprising results both for physics and mathematics applications.

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