MODEL OF A LINEAR HARMONIC OSCILLATOR WITH A POSITION-DEPENDENT MASS IN THE EXTERNAL HOMOGENEOUS FIELD. THE CASE OF A PARABOLIC WELL

SHAKIR M. NAGIYEV, SHOVQIYYA A. AMIROVA
Institute of Physics, National Academy of Sciences of Azerbaijan
H. Javid ave.131, AZ1143 Baku, Azerbaijan

An exactly solvable model of a linear harmonic oscillator with position-dependent mass in the presence of an external uniform field is constructed. The interaction potential is an infinite parabolic well. It is shown that the system has only a discrete energy spectrum, and the number of levels is finite and depends on the modulus and sign of the force. The wave functions are expressed in terms of Laguerre polynomials.

Keywords: Harmonic oscillator; external homogeneous field; position-dependent mass.
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1. INTRODUCTION

For several decades now, various quantum mechanical systems with a position-dependent *mass* *M(x)* have been intensively studied by many authors [1-25]. The great interest of physicists in such systems is explained by the fact that these systems play an important role in many physical problems. For example, they are widely used in condensed matter physics, in materials science, in nuclear physics, etc. They have found particular application in the study of the electronic properties of semiconductors [6], in the theories of quantum dots, quantum wells [7], [24] and quantum liquids [25], in the nuclear many-body problem [11], etc.

Exactly solvable problems occupy a special place among the problems of quantum mechanics. The Schrödinger wave equation completely describes the dynamical behavior of nonrelativistic microscopic systems. However, there are only a very limited number of potentials important for physical applications that allow exact analytical solutions of the Schrödinger equation. It is well known that the exact analytical solution of the Schrödinger equation for a given quantum system provides the maximum possible information about this system. On the other hand, finding exact solutions of the Schrödinger equation with position-dependent mass turns out to be very useful for understanding some physical phenomena and testing some approximation methods. In this regard, we note that a number of researches [26–44] are devoted to the construction of exactly solvable potentials for the Schrödinger equation with the position-dependent mass.

In a recent paper [45], we constructed a new exactly solvable model of a linear quantum harmonic oscillator with a position-dependent mass whose interaction potential behaves like a semi-restricted quantum well with a non-rectangular profile. The frequency of the constructed oscillator model is chosen as a constant value.

The purpose of this paper is to construct, on the basis of the model in [45], a new model of an oscillator with position-dependent mass $M(x)$ and frequency $\omega(x)$ so that the stiffness coefficient of the oscillator remains constant: $k = M(x)\omega^2(x) = \text{const.}$

We emphasize that the construction of models of quantum physical systems with the position-dependent mass starts with choosing the form of the free Hamiltonian H_0 and the subsequent selection of the mass function $M(x)$. The point is that due to the non-commutativity of the momentum operator $\hat{p} = -i\hbar\partial_x$ and the mass function $M(x)$, the question arises of their ordering in the expression for the free non Hermitian Hamiltonian.

We structured our paper as follows. Section 2 presents a brief review of the exact solution of the usual nonrelativistic quantum harmonic oscillator with the constant mass m_0 approach. Section 3 consists of a brief discussion of the property of the generalized Hamiltonian [33, 34]. Section 4 devoted to the construction of the exactly solvable model of the nonrelativistic harmonic oscillator model with a position-dependent effective mass and frequency, interaction potential of which behaves itself as a symmetric infinite parabolic quantum well. Section 4 is devoted to building a model of a nonrelativistic harmonic oscillator with position-dependent mass and frequency, so that the interaction potential is a symmetric infinite parabolic quantum well.

In final section 5, we discuss the limit cases, when the parameter a goes to infinity. As a consequence, the wave functions of the model under construction expressed by the Laguerre polynomials completely recover the wave functions of the non-relativistic quantum harmonic oscillator with constant mass and frequency as well as energy spectrum of non-equidistant and finite form becomes equidistant and infinite.

2. NONRELATIVISTIC LINEAR HARMONIC OSCILLATOR WITH CONSTANT MASS AND FREQUENCY

Let us write the one-dimensional Schrödinger equation describing the motion of a nonrelativistic quantum particle with constant mass m_0 in the external field $V(x)$. It has the form

131, H.Javid ave, AZ-1143, Baku
ANAS, Institute of Physics
E-mail: jophphysics@gmail.com

$$\left[\frac{\hat{p}^2}{2m_0}+V(x)\right]\psi(x)=E\psi(x), \qquad (2.1)$$

where $\hat{p}=-i\hbar\partial_x$ is the momentum operator. A linear harmonic oscillator with frequency ω_0 corresponds to the following potential energy

$$V^{\mathrm{HO}}(x)=\frac{m_0\omega^2x^2}{2}, \quad -\infty<x<\infty, \qquad (2.2)$$

Let us rewrite equation (2.1) with the potential (2.2) as

$$\frac{d^2\psi}{dx^2}+\frac{2m_0}{\hbar^2}\left(E-\frac{m_0\omega^2x^2}{2}\right)\psi=0. \qquad (2.3)$$

The solution and energy spectrum of the equation (2.3) are well known [45]

$$\psi_n^{\mathrm{HO}}(x)=c_n^{\mathrm{HO}}e^{-\frac{1}{2}\lambda_0^2x^2}H_n(\lambda_0x), \qquad (2.4)$$

$$E_n^{\mathrm{HO}}=\hbar\omega\left(n+\frac{1}{2}\right),\ n=0,1,2,3,\ldots, \qquad (2.5)$$

Where $H_n(x)$ are Hermite polynomials and $\lambda_0=\sqrt{m_0\omega_0/\hbar}$. Normalization constants are equal to

$$c_n^{\mathrm{HO}}=\sqrt{\frac{\lambda_0}{2^nn!\sqrt{\pi}}}=\frac{1}{\sqrt{2^nn!}}\left(\frac{m_0\omega_0}{\pi\hbar}\right)^{1/4}. \qquad (2.6)$$

They are found from the orthogonality condition for the Hermite polynomials [46, 47]

$$\int_{-\infty}^{\infty}e^{-x^2}H_m(x)H_n(x)dx=\sqrt{\pi}2^nn!\,\delta_{nm}. \qquad (2.7)$$

3. ON THE GENERALIZED FREE HAMILTONIAN WITH THE POSITION-DEPENDENT MASS

We emphasize that the construction of models of quantum physical systems with the position-dependent mass starts with choosing the form of the free Hamiltonian H_0 and the subsequent selection of the mass function $M(x)$. A long-standing problem with the position-dependent mass model is how to order the ambiguities that appear due to the non-commutativity of the momentum operators $\hat{p}=-i\hbar\partial_x$and the mass $M(x)$ in the expression for the free Hamiltonian. In [33, 34], a Hamiltonian with a position-dependent mass was proposed, which can be represented as

$$H_0=\frac{1}{2}\hat{p}\frac{1}{M(x)}\hat{p}+V_{\mathrm{free}}(x), \qquad (3.1)$$

where $V_{\mathrm{free}}(x)$ is the contribution from the free Hamiltonian to the potential energy, which depends on the mass function $M(x)$ and on the $3N$ ordering parameters $\alpha_i,\gamma_i\ (i=1,2,\ldots,N),\ N=1,2,3,\ldots$ It has a form

$$V_{\mathrm{free}}(x)=A_f\frac{\hbar^2M'^2}{2M^3}-B_f\frac{\hbar^2M''}{4M^2}. \qquad (3.2)$$

Here $A_f=\overline{\alpha}+\overline{\gamma}+\overline{\alpha\gamma}$, $B_f=\overline{\alpha}+\overline{\gamma}$ and $\overline{\alpha},\overline{\gamma}$ are the mean values of the ordering parameters

$$\overline{\alpha}=\frac{1}{N}\sum_{i=1}^N\alpha_i\,,\overline{\gamma}=\frac{1}{N}\sum_{i=1}^N\gamma_i,\overline{\alpha\gamma}=\frac{1}{N}\sum_{i=1}^N\alpha_i\gamma_i. \qquad (3.3)$$

We emphasize that the parameters A_f and B_f can take any real values. Hamiltonian (3.1) can be written as the arithmetic mean of the von Roos Hamiltonian, i.e.

$$H_0=\frac{1}{N}\sum_{i=1}^N H_{0i}^{\mathrm{vR}},$$
$$H_{0i}^{\mathrm{vR}}=\frac{1}{4}\left(M^{\alpha_i}\hat{p}M^{\beta_i}\hat{p}M^{\gamma_i}+M^{\gamma_i}\hat{p}M^{\beta_i}\hat{p}M^{\alpha_i}\right). \qquad (3.4)$$

The sum (3.4) contains all terms with equal weights (probabilities) $1/N$. If we assume that they enter the sum with different weights θ_i/N, then instead of (3.1) we obtain the following Hamiltonian

$$H_0^\theta=\frac{1}{N}\sum_{i=1}^N\theta_iH_{0i}^{\mathrm{vR}}=\frac{1}{2}\hat{p}\frac{1}{M(x)}\hat{p}+\mathrm{V}_{\mathrm{free}}^\theta(x), \qquad (3.5)$$

where

$$\frac{1}{N}\sum_{i=1}^N\theta_i=1,\ \mathrm{V}_{\mathrm{free}}^\theta(x)=A_f^\theta\frac{\hbar^2M'^2}{2M^3}-B_f^\theta\frac{\hbar^2M''}{4M^2}. \qquad (3.6)$$

For the coefficients A_f^θ and B_f^θwe have the expressions

$$A_f^\theta=\frac{1}{N}\sum_{i=1}^N\theta_i(\alpha_i+\gamma_i+\alpha_i\gamma_i),\ B_f^\theta=\frac{1}{N}\sum_{i=1}^N\theta_i(\alpha_i+\gamma_i). \qquad (3.7)$$

A different choice of the values of the parameters A_f^θ and B_f^θ generates different Hamiltonians. On the other hand, different choices of the values of the parameters $N,\ \theta_i,\ \alpha_i,\ \gamma_i\ \ (i=1,2,\ldots,N)$ can correspond to the same values of the parameters A_f^θ and B_f^θ. Then all these Hamiltonians will be physically equivalent to each other. Thus, the form of the Hamiltonian with a position-dependent mass is determined by only two parameters A_f^θ and B_f^θ, i.e. it turns out that the Hamiltonians (3.1) and (3.5) are physically equivalent and differ from each other only by renaming.

As an example, we list various orderings that can be found in the literature and give the value of the coefficients for them A_f^θ and B_f^θ.

1. Ben-Daniel–Duke ordering [1]: $\alpha=\gamma=0,\beta=-1,\ A_f=B_f=0$;
2. Gore–Williams ordering [3]: $\alpha=-1,\beta=\gamma=0,\ A_f=B_f=-1$;

3. Zu-Kremer ordering [4]: $\alpha = \gamma = -1/2$,$\beta = 0$, $A_f = -3/4$, $B_f = -1$;
4. Li-Kun ordering [5]: α =0, $\beta = \gamma = -1/2$ 0, $A_f = B_f = -1/2$;
5. Mustafa–Mazharimusavi ordering [15]: $\alpha = \gamma = -1/4, \beta = -1/2$. $A_f = -7/16$, $B_f = -1/2$.

4. A MODEL OF LINEAR HARMONIC OSCILLATOR WITH THE POSITION-DEPENDENT MASS AND FREQUENCY: PARABOLIC WELL

In a recent paper [34], we constructed an exactly solvable model of a nonrelativistic linear quantum harmonic oscillator with a position-dependent mass. The dependence of the mass on the position is chosen in the form

$$M(x) = \frac{a^2 m_0}{a^2+x^2}, \quad x+a>0, \quad a>0. \quad (4.1)$$

The potential energy of the oscillator in [34] has the form

$$\mathrm{V}^{\mathrm{HO}}(x) = \begin{cases} \frac{M(x)\omega_0^2 x^2}{2}, & \mathrm{x+a}>0, \\ \infty, & \mathrm{x+a} \le 0 \end{cases} \quad (4.2)$$

and behaves like a semi-infinite quantum well. The oscillator frequency is constant, i.e. $\omega_0 = \mathrm{const}$. This model is described by the free Hamiltonian Ben Daniel-Duke

$$H_0^{\mathrm{BD}} = -\hbar^2 \partial_{\mathrm{x}} \frac{1}{2M(x)} \partial_{\mathrm{x}}. \quad (4.3)$$

The wave functions of the discrete spectrum in this case are expressed in terms of the Bessel polynomials$y_n(x;\alpha)$:

$$\Psi_n^{QW}(x) = \mathrm{const}\left(1+\frac{x}{\alpha}\right)^{-\lambda_0^2 a^2} e^{-\frac{\lambda_0^2 a^2}{a+x}} y_n\left(\frac{x+a}{\lambda_0^2 a^3}; -2\lambda_0^2 a^2\right). \quad (4.4)$$

Discrete energy spectrum is not equidistant

$$E_n^{QW} = \hbar\omega_0\left(n+\frac{1}{2}\right) - \frac{\hbar^2}{2m_0 a^2} n(n+1), \ n = 0,1,2,\dots,N. \quad (4.5)$$

Since the quantum well (3.2) has a finite depth, the number of energy levels of the discrete spectrum is finite.

The purpose of this section is to construct a new model of a linear harmonic oscillator with a mass function of the form (4.1) and with the following interaction potential

$$\mathrm{V}(x) = \begin{cases} \frac{m_0\omega_0^2 x^2}{2} + gx, & \mathrm{x+a}>0, \\ \infty, & \mathrm{x+a} \le 0. \end{cases} \quad (4.6)$$

Thus, we assume that the frequency $\omega(x)$ of the considered model also depends in a certain way on the position

$$\omega(x) = \omega_0\left(1+\frac{x}{a}\right). \quad (4.7)$$

Potential (4.6) corresponds to an asymmetric infinite semi-parabolic well (Fig. 3).

To describe the oscillator model under consideration, we will use the generalized Hamiltonian (3.1). In the case of the mass function (4.1), the free potential $V_{free}(x)$) becomes constant, i.e.

$$V_{free}(x) = \frac{\hbar^2}{2a^2 m_0}\left(4A_f - 3B_f\right) \equiv V_0 = \mathrm{const}. \quad (4.8)$$

For our oscillator model, the Schrödinger equation reads:

$$\psi'' + \frac{2}{a+x}\psi' + \frac{2a^2 m_0}{\hbar^2(a+x)^2}\left(\varepsilon - \frac{m_0\omega_0^2 x^2}{2} - gx\right)\psi = 0, \quad (4.9)$$

where

Where $\varepsilon = E - V_0$. By introducing a new variable $\xi = x/a$ $(-1<\xi<\infty)$ we rewrite equation (4.9) in the following form

$$a_2\psi'' + a_1\psi' + a_0\psi = 0. \quad (4.10)$$

Here we introduced the notation

$$a_2 = 1, \ a_1 = \frac{2}{a+x}, \ a_0 = \frac{c_0 - c_1\xi - c_2\xi^2}{(1+\xi)^2}. \quad (4.11)$$

For the coefficients c_0, c_1 and c_2 we have the following expressions

$$c_0 = \frac{2a^2 m_0\varepsilon}{\hbar^2}, c_2 = \frac{m_0\omega_0^2 a^4}{\hbar^2} = \lambda_0^4 a^4, \ c_1 = \frac{2m_0 a^3 g}{\hbar^2}. \quad (4.12)$$

The solution of equation (4.10) will be sought in the form

$$\psi = \varphi(\xi)y(\xi), \ \ \varphi(\xi) = (1+\xi)^A e^{-B(1+\xi)}. \quad (4.13)$$

The function $\varphi(\xi)$must be finite for all finite values of $\xi(-1<\xi<\infty)$. It follows from this requirement that there must be $\mathrm{A} \ge 0, B>0$.

If we substitute (4.13) into equation (4.10), then we obtain for the function $y(\xi)$ an equation of the form

$$b_2 y'' + b_1 y' + b_0 y = 0, \quad (4.14)$$

$$b_2 = a_2, b_1 = a_1 + 2a_2\frac{\varphi'}{\varphi}, \ b_0 = a_0 + a_1\frac{\varphi'}{\varphi} + a_2\frac{\varphi''}{\varphi}. \quad (4.15)$$

After simple calculations, from here we find

$$b_2 = 1, b_1 = \frac{2(1+A-B-B\xi)}{1+\xi}, b_0 = \frac{\sigma}{(1+\xi)^2}. \tag{4.16}$$

Here σ is a square nominal $\sigma = \alpha_2\xi^2 + \alpha_1\xi + \alpha_0$ with coefficients

$$\alpha_2 = B^2 - c_2,\ \alpha_1 = -c_1 - 2B - 2AB + 2B^2\ \alpha_0 = c_0 + A - 2B + (A-B)^2. \tag{4.17}$$

We find the unknown parameters A and B in (4.13) from the condition that the square trinomial σ is divisible by $1+\xi$, т.е. $\sigma = \mu(1+\xi)$. This condition gives

$$\alpha_2 = 0,\ \alpha_1 = \alpha_0 = \mu. \tag{4.18}$$

From equations (4.18) we find the unknown parameters A, B and μ. They are equal

$$A = -\frac{1}{2} + \sqrt{\frac{1}{4} + B^2 - c_0 - c_1}\ ,\ B = \lambda_0^2 a^2,$$

$$\mu = 2B^2 - 2B - 2AB - c_1. \tag{4.19}$$

If we now take into account the conditions $A \geq 0$, then we get that the coefficient $c_0$is bounded from above, i.e.

$$c_0 \leq \lambda_0^4 a^4 - c_1. \tag{4.20}$$

This results in an upper bound on the energy value:

$$\varepsilon \leq \frac{m_0\omega_0^2 a^2}{2} - ag\ . \tag{4.21}$$

Thus, despite the fact that the potential well of our oscillator model (4.6) has an infinite depth, the number of energy levels of the discrete spectrum will be finite.

Taking into account (4.19), we rewrite equation (4.14) in the form

$$(1+\xi)y'' + (2 + 2A - 2B - 2B\xi)y' + \mu\, y = 0. \tag{4.22}$$

By substituting $z = 2B(1+\xi)$ this equation is reduced to the equation for the degenerate hypergeometric function [45]

$$zu'' + (\gamma - z)u' - \alpha u = 0\ ,\ u = {}_1F_1(\alpha; \gamma, z), \tag{4.23}$$

where

$$\gamma = 2(1+A),\ \alpha = A + 1 - B + \frac{c_1}{2B}. \tag{4.24}$$

The general solution of equation (4.22) has the form

$$y = C_1\ {}_1F_1(\alpha; \gamma, z) + C_2 z^{1-\gamma}\ {}_1F_1(\alpha - \gamma + 1; 2 - \gamma, z). \tag{4.25}$$

Since $1 - \gamma = -1 - A < 0$, then the second term in (4.25) diverges at the point $\xi = -1$. It follows that $C_2 = 0$. Then for the function (4.25) we will have the expression

$$y = C_1\ {}_1F_1\left(A + 1 - B + \frac{c_1}{2B}; 2 + 2A; 2B(1+\xi)\right). \tag{4.26}$$

The function $y(\xi)$ (4.26) must be finite for all finite ξ, and for $\xi = \infty$ it can go to infinity no faster than a finite power of ξ (so that the function ψ (4.13) goes to zero). This implies that the function y (4.26) can only be a polynomial in ξ. For this, the condition will have to be fulfilled (the condition for energy quantization)

$$A + 1 - B + \frac{C_1}{2B} = -n,\ n = 0,1,2,\ldots \tag{4.27}$$

Substituting (4.27) into (4.26) and taking into account the definition of Laguerre polynomials [46,47]

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\ {}_1F_1(-n; \alpha+1; x), \tag{4.28}$$

we conclude that the function $y(\xi)$ is expressed in terms of the Laguerre polynomials

$$y(\xi) = L_n^{2A_n+1}\big(2B(1+\xi)\big). \tag{4.29}$$

Then the complete wave function (4.13) will have the form

$$\psi_n(\xi) = C_n(1+\xi)^{A_n} e^{-B(1+\xi)} L_n^{2A_n+1}\big(2B(1+\xi)\big), \tag{4.30a}$$

or

$$\psi_n(x) = C_n(a+x)^{\lambda_0^2 a^2 - n - 1 - \frac{ag}{\hbar\omega_0}} e^{-\lambda_0^2 a(a+x)} L_n^{2\lambda_0^2 a^2 - 1 - 2n - \frac{2ag}{\hbar\omega_0}}\left(2\lambda_0^2 a(a+x)\right). \qquad (4.30b)$$

Corresponding to the wave functions (4.30), the energy spectrum is discrete

$$E_n = \hbar\Omega_0\left(n+\frac{1}{2}\right) - \frac{\hbar^2}{2m_0a^2}n(n+1) - \frac{m_0\omega_0^2x_0^2}{2} + V_0, \qquad (4.31)$$

where $\Omega_0 = \omega_0\left(1-\frac{x_0}{a}\right)$ is a renormalized frequency of the oscillator model under consideration and $x_0 = g/(m_0\omega_0^2$. As follows from here that the number of energy levels N_g is finite and depends on both the value and sign of the force, i.e.

$$n = 0,1,2,\dots,N_g,\ N_g = \lambda_0^2a^2 - 1 - \frac{m_0\omega_0 a x_0}{\hbar}. \quad (4.32)$$

Depending on the sign of the force g (or x_0), the frequency of the oscillator or can increase or decrease. For $g > 0$, with an increase in the force modulus, the number of levels increases, and for $g < 0$, on the contrary, it decreases. In the second case, even at a certain force value equal to $g = m_0\omega_0^2 a - \hbar\omega_0/a$, the spectrum disappears altogether.

In this paper, we have constructed an exactly solvable model of a linear quantum harmonic oscillator with a position-dependent mass. The model under consideration has only discrete energy spectrum, and despite the fact that the potential parabolic well has an infinite depth, the number of levels is finite. This can be explained by the influence of the mass function $M(x)$ on the potential. In other words, it is possible that the considered model with a position-dependent mass and with an infinite well is equivalent to a system with a constant mass and a well of finite depth.

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